The University of Sheffield

Department of Automatic Control & Systems Engineering

Linearization Methods and Control of Nonlinear Systems – Two Cases

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Importance of studying Nonlinear Systems

- No model of a real system is truly linear even if it is profitable to study their linear approximation.
- Nonlinear systems may show complex effects (chaos, bifurcations, etc) that cannot be anticipated.
- It brings many disciplines together: mathematics, physics, biology, chemistry, engineering, economics, medicine, etc.

Importance of Linearization Methods

“They allow to adopt linear control techniques to analyze nonlinear problems”

Some Linearization Techniques

- Jacobian Linearization
- **Carleman Linearization**
- Linearization using Lie Series
- **Iteration Technique**
- Feedback linearization
- Linearization via changes of variables

Carleman Linearization and Lyapunov Stability Theory

Navarro Hernandez, C. and Banks, S.P. (2004), A Generalization of Lyapunov's Equation to Nonlinear Systems, *NOLCOS 2004*, Stuttgart, Germany, September 2004

Banks, S.P. and Navarro Hernandez, C. (2005), A New Proof of McCann's Theorem and the Generalization of Lyapunov's Equation to Nonlinear Systems, *IJICIC*, to appear.

Carleman Linearization

Nonlinear System → Infinite Dimensional Linear System

- Kowalski, K. and Steeb, W. *Nonlinear dynamical systems and Carleman linearization*. World Scientific Publishing Co. Singapore, 1991.

- Gaude, Brian. *Solving nonlinear aeronautical problems using the Carleman Linearization Method*. Sandia National Laboratories Report, U.S.A, 2001.

Applications to Control

- **Domains of attraction of nonlinear feedback systems** (Loparo K.A. and Blankenship (1978) Estimating the domain of attraction of nonlinear feedback systems. *IEEE Trans. Aut. Control*, **AC-23**, 602)
- **Design of observers with linear error dynamics** (Deutscher, J. (2003) Asymptotically exact input-output linearization using Carleman linearization. *ECC2003* Cambridge, UK)
- **Solutions of Lotka-Volterra models** *(Steeb and Wilhelm, 1980)*
- **Power series expansions for nonlinear systems** *(Brenig and Fairén, 1981)*
- **Construction of approximate Monte-Carlo-like solutions to nonlinear integral equations** *(Ermakov, 1984)*
- **Study of nonlinear partial differential equations** *(Kowalski, 1988)*

Carleman Linearization

Consider the following nonlinear system

$$\dot{x} = f(x)$$

the Taylor series can be written as

$$f(x) = \sum_{\ell} A^{\ell} x^{[\ell]}$$

where

$$A^{\ell} = \left\{ \frac{1}{\ell!} \frac{\partial^{\ell} f(0)}{\partial x_{i_1} \cdots \partial x_{i_\ell}} \right\} \qquad x^{[\ell]} = \underbrace{x \otimes x \otimes \ldots \otimes x}_{\ell-times}$$

To obtain the linear system

$$x^{[k]} = \underbrace{x \otimes x \otimes \ldots \otimes x}_{k-times}$$

$$\dot{x}^{[k]} = \sum_{i=1}^{k} x \otimes \ldots \otimes \underbrace{\dot{x}}_{i} \otimes \ldots \otimes x$$

$$= \sum_{i=1}^{k} x \otimes \ldots \otimes f(x) \otimes \ldots \otimes x$$

$$\dot{x}^{[k]} = \sum_{i=1}^{k} x \otimes \ldots \otimes \sum_{\ell} A^{\ell} x^{[\ell]} \otimes \ldots \otimes x$$

Therefore

$$\dot{x}^{[k]} = \sum_{\ell} A_k^{\ell} x^{[k+\ell-1]}$$

$$A_k^{\ell} = \sum_{i=1}^{k} I \otimes \ldots \otimes I \otimes A^{\ell} \otimes I \otimes \ldots \otimes I$$

$$A_k^{\ell} = A_1^{\ell} \otimes I^{[k-1]} + I \otimes A_{k-1}^{\ell}$$

Setting

$$w = (x, x \otimes x, x \otimes x \otimes x, \cdots)^T$$

We obtain the infinite linear system

$$\dot{w} = \mathrm{A}w$$

$$\mathrm{A} = \begin{pmatrix} A_1^1 & A_1^2 & A_1^3 & \cdots & \cdots & \cdots \\ 0 & A_2^1 & A_2^2 & A_2^3 & \cdots & \cdots \\ 0 & 0 & A_3^1 & A_3^2 & A_3^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Lyapunov's Stability

$V(x)$ is a Lyapunov function for the equilibrium x^*

if

1. $V(x) > 0$
2. $\dot{V}(x) < 0$

$\forall x \neq x^*$

Problem → Find the Lyapunov function

Lyapunov's Equation

Linear time invariant system

$$\dot{x} = Ax$$

is asymptotically stable iff

$$A^T P + PA = -Q$$

The Lyapunov function is

$$V = x^T P x$$

- Vanelli, A. and Vidyasagar, M. (1985). Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems, *Automatica*, **21**, 69-80. (Lyapunov functions that are rational rather than polynomial)
- Camilli, F., Grüne L. and Wirth, F. Zubov´s method for perturbed differential equations. NOLCOS 2004. Stuttgart, Germany (Generation of Lyapunov functions and domains of attraction using Zubov method)

Lyapunov's Equation and McCann's Theorem

Consider the linear system $\dot{w} = \mathrm{A}w$

and let P satisfy the Lyapunov equation

$$\mathrm{A}^T\mathrm{P} + \mathrm{PA} = -\mathrm{I}$$

Then,

$$V = \left\langle (x, x \otimes x, x \otimes x \otimes x, \ldots), \mathrm{P}(x, x \otimes x, x \otimes x \otimes x, \ldots) \right\rangle$$

$$\dot{V} = -\left\| (x, x \otimes x, x \otimes x \otimes x \ldots) \right\|^2 = -\left(e^{\|x\|^2} - 1 \right)$$

The nonlinear system

$$\dot{x} = f(x)$$

is globally asymptotically stable iff

$$A^T P + PA = -I \qquad (*)$$

is soluble for P (positive definite) .

McCann's Theorem - Global asymptotically stable dynamical systems are equivalent to linear systems. (McCann, 1979)

If $\dot{x} = f(x)$ ***is globally asymptotically stable there is a positive definite solution of*** (*)

Example #1

Consider the nonlinear system $\dot{x} = -x + x^3$

and obtain the infinite linear system $\dot{w} = \mathrm{A}w$

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & -2 & 0 & 2 & 0 & \cdots & \cdots \\ 0 & 0 & -3 & 0 & 3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$w = (x, x^2, x^3, x^4, \ldots)^T$$

Truncate A and solve Lyapunov Equation

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}\mathbf{P} + \mathbf{P}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} = -\mathbf{I}$$

Obtain Lyapunov function

$$V = \tfrac{1}{2}x^2 + \tfrac{1}{2}x^4 + \tfrac{5}{24}x^6$$

$$\dot{V} = (x^2 - 1)(x^2 + 2x^4 + \tfrac{5}{4}x^6)$$

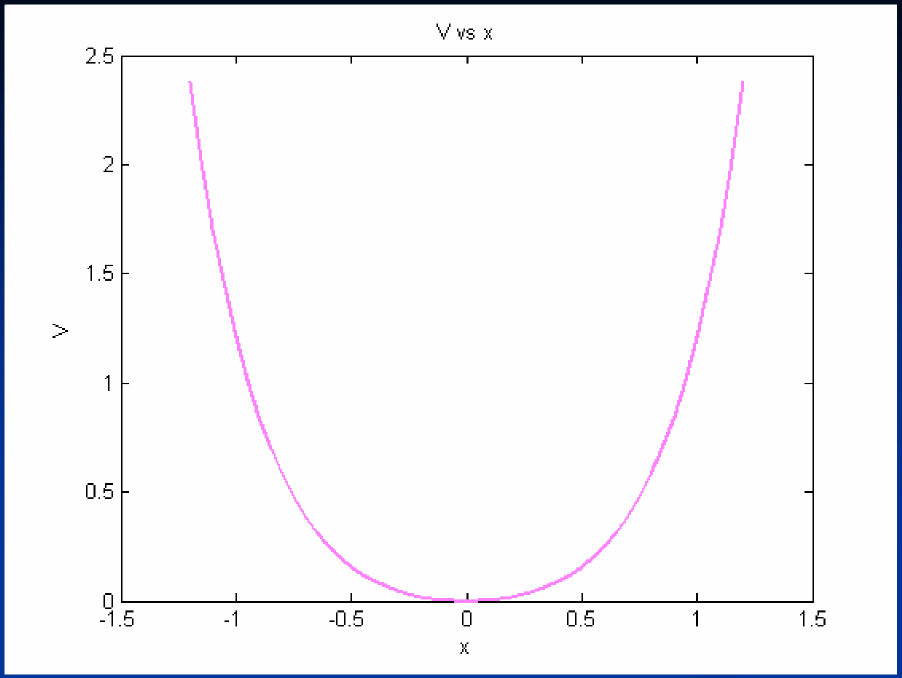
Truncating A to a 7x 7 matrix

$$V = \frac{1}{2}x^2 + \frac{1}{2}x^4 + \frac{1}{2}x^6 + \frac{1}{2}x^8 + \frac{253}{640}x^{10} + \frac{1087}{3840}x^{12} + \frac{1343}{10752}x^{14}$$

$$\dot{V} = (x^2 - 1)(x^2 + \frac{7680}{3840}x^4 + \frac{11520}{3840}x^6 + \frac{15360}{3840}x^8 + \frac{15180}{3840}x^{10} + \frac{13044}{3840}x^{12} + \frac{6715}{3840}x^{14})$$

Fig. 1 $V(x)$ vs x

$$V = \tfrac{1}{2}x^2 + \tfrac{1}{2}x^4 + \tfrac{5}{24}x^6$$



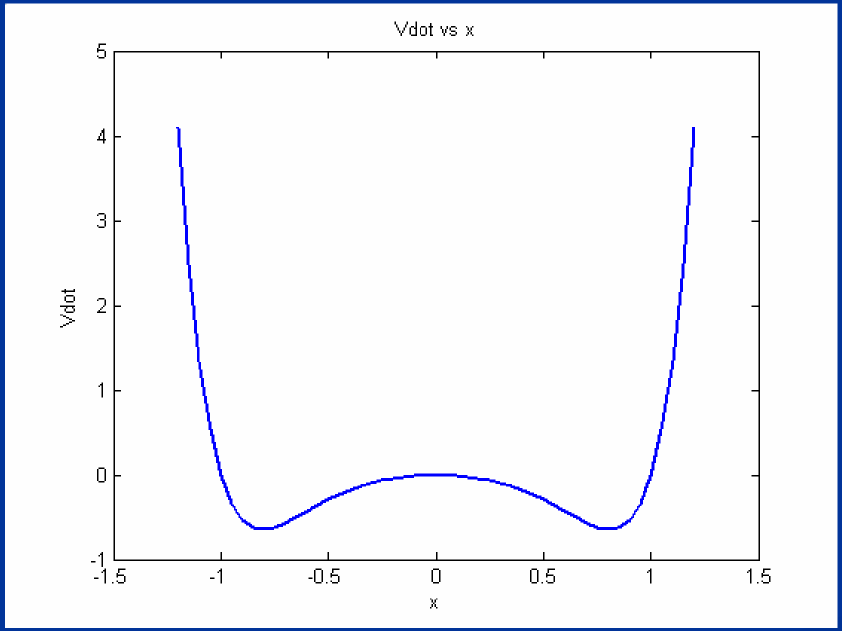


Fig. 2 $\dot{V}(x)$ vs x

$$\dot{V} = (x^2 - 1)(x^2 + 2x^4 + \tfrac{5}{4}x^6)$$

Example #2

Consider the nonlinear system

$$\dot{x} = -x + 4y$$
$$\dot{y} = -x - y^3$$

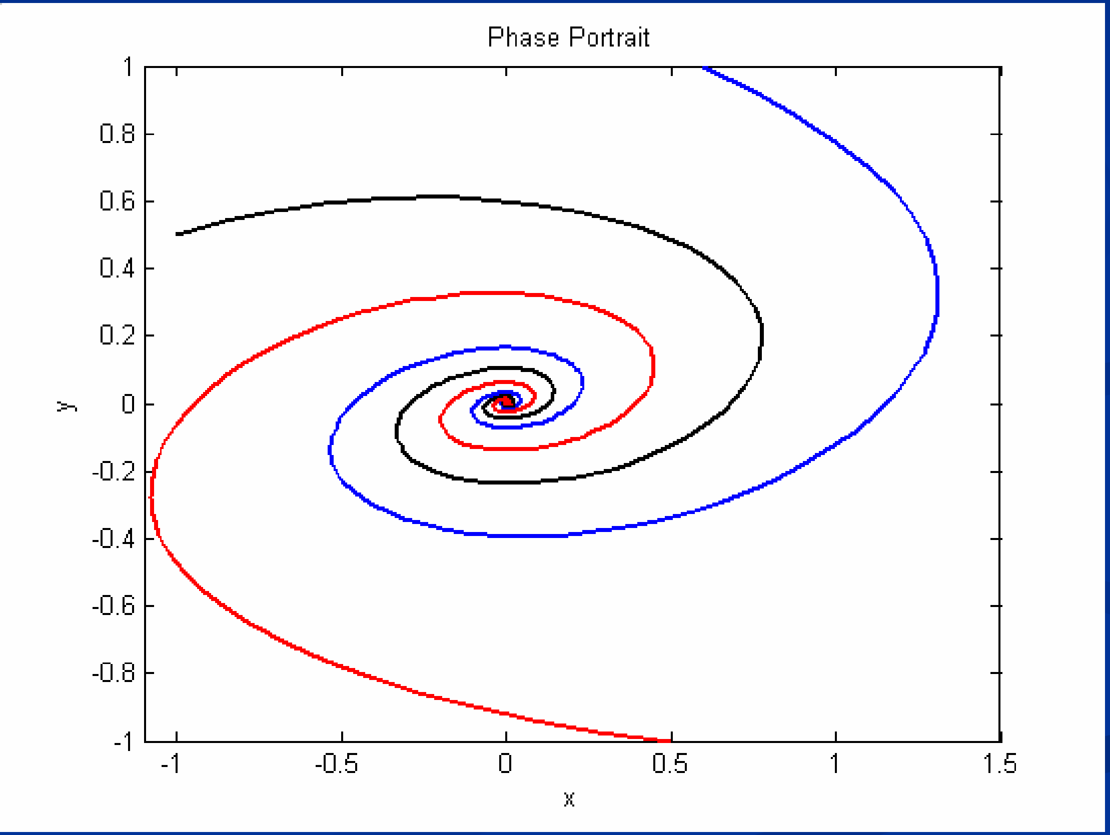


Fig. 3 Phase Portrait

Nonlinear System:

$$\dot{x} = -x + 4y$$
$$\dot{y} = -x - y^3$$

Now

$$w = (\underbrace{x, y}_{\mathbf{x}}, \underbrace{x^2, xy, yx, y^2}_{\mathbf{x}\otimes\mathbf{x}}, \underbrace{x^3, x^2y, x^2y, xy^2, x^2y, xy^2, xy^2, y^3}_{\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}}, \ldots)^T$$

$$\mathrm{A} = \begin{pmatrix} A_1^1 & A_1^2 & A_1^3 & 0 & 0 & 0 \\ 0 & A_2^1 & A_2^2 & A_2^3 & 0 & 0 \\ 0 & 0 & A_3^1 & A_3^2 & A_3^3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$A_1^1 = \begin{pmatrix} -1 & 4 \\ -1 & 0 \end{pmatrix} \quad A_1^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_1^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

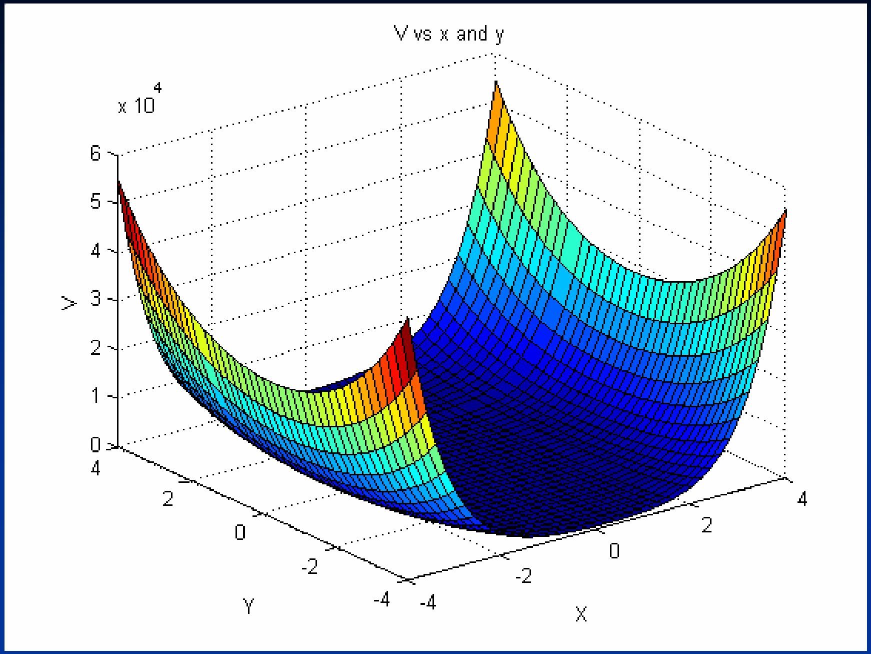


Fig. 4 *V(x,y)* vs *x,y*

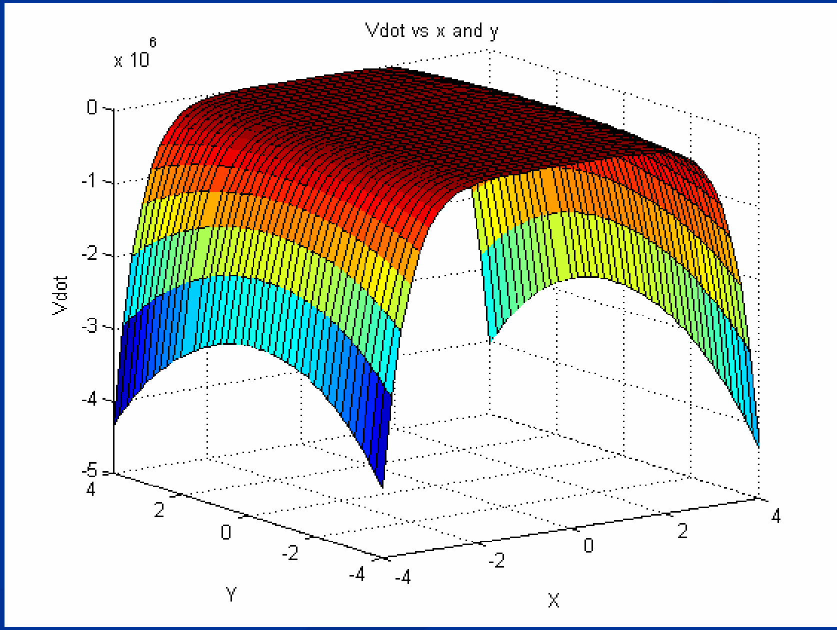


Fig. 5 $\dot{V}(x,y)$ vs x,y

Conclusions

- A method to find a Lyapunov function for stable nonlinear systems by using a defined Lyapunov equation.
- By increasing the number of terms of the truncation of the infinite-dimensional Lyapunov equation it is expected that the method will approximate in a better way the basin of attraction of the systems.
- It is computationally difficult as the operator A grows exponentially and the Lyapunov expressions are complicated

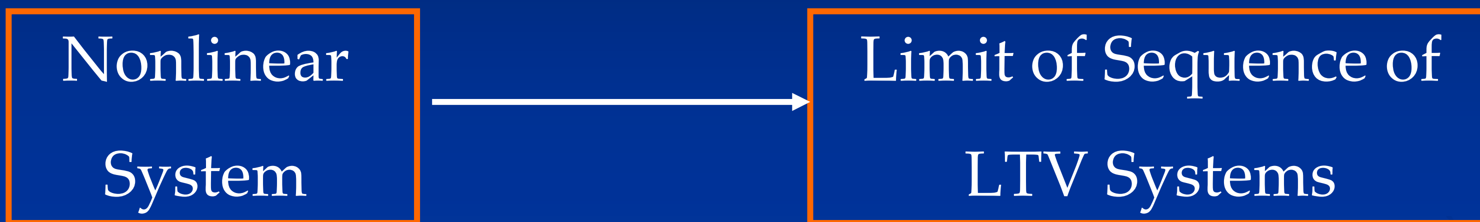
Future Work

- Study of the required level of truncation of the Lyapunov equation
- Use of the Lyapunov functions for Control Design by introducing unknown parameters in the operator **A**
- Definition of the Lyapunov function by expanding the nonlinear system using orthogonal functions

Iteration Technique and Fault Detection for Nonlinear Systems

Navarro Hernandez, C, Crusca, F., Aldeen, M. and Banks, S. P. (2005), Fault Detection for Nonlinear Systems using Linear Approximations, *To be presented at IFAC2005*, Prague, Cz. July 2005

Iteration Technique



- Tomas-Rodriguez, M., Banks, S., (2003) Linear approximations to nonlinear dynamical systems with applications to stability and spectral theory, *IMA Journal of Mathematical Control and Information*, **20**, 89-103.

Applications to Control

- **Stability and spectral theory**

  Tomas-Rodriguez M. and Banks, S., (2003) Linear approximations to nonlinear dynamical systems with applications to stability and spectral theory, *IMA Journal of Mathematical Control and Information*, **20**, 89-103.

- **Design of Observers**

  Navarro Hernandez, C., Banks, S.P. and Aldeen, M. (2003) Observer design for nonlinear systems using linear approximations, *IMA Journal of Mathematical Control and Information*, **20**, 359-370.

- **Pole Placement for Nonlinear Systems**

  Tomas-Rodriguez M. and Banks, S.P. (2004). Pole placement for non-linear systems., *NOLCOS 2004*, Stuttgart, Germany.

- **Optimal Control**

  Çimen, T and Banks, S.P. (2004). Nonlinear optimal tracking control with application to super-tankers for autopilot design, *Automatica*.

Having the nonlinear system

$$\dot{x} = A(x)x, x(0) = x_0 \in \Re^n.$$

and introducing the sequence of linear time varying equations:

$$\dot{x}^{[1]}(t) = A(x_0)x^{[1]}(t), x^{[1]}(0) = x_0$$
$$\vdots$$
$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t), x^{[i]}(0) = x_0$$

where i=number of approximations, it can be shown that the solution of this sequence converges to the solution of the original nonlinear system if the Lipschitz condition is statisfied.

Lipschitz condition $\|A(x) - A(y)\| \le \alpha\|x - y\|$

Solution to Van der Pol oscillator

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1^{\ 2}+1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and for the ith approximation,

$$\begin{pmatrix} \dot{x}_1^{[i]}(t) \\ \dot{x}_2^{[i]}(t) \end{pmatrix} = \begin{pmatrix} -(x_1^{[i-1]}(t))^2+1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1^{[i]}(t) \\ x_2^{[i]}(t) \end{pmatrix}$$

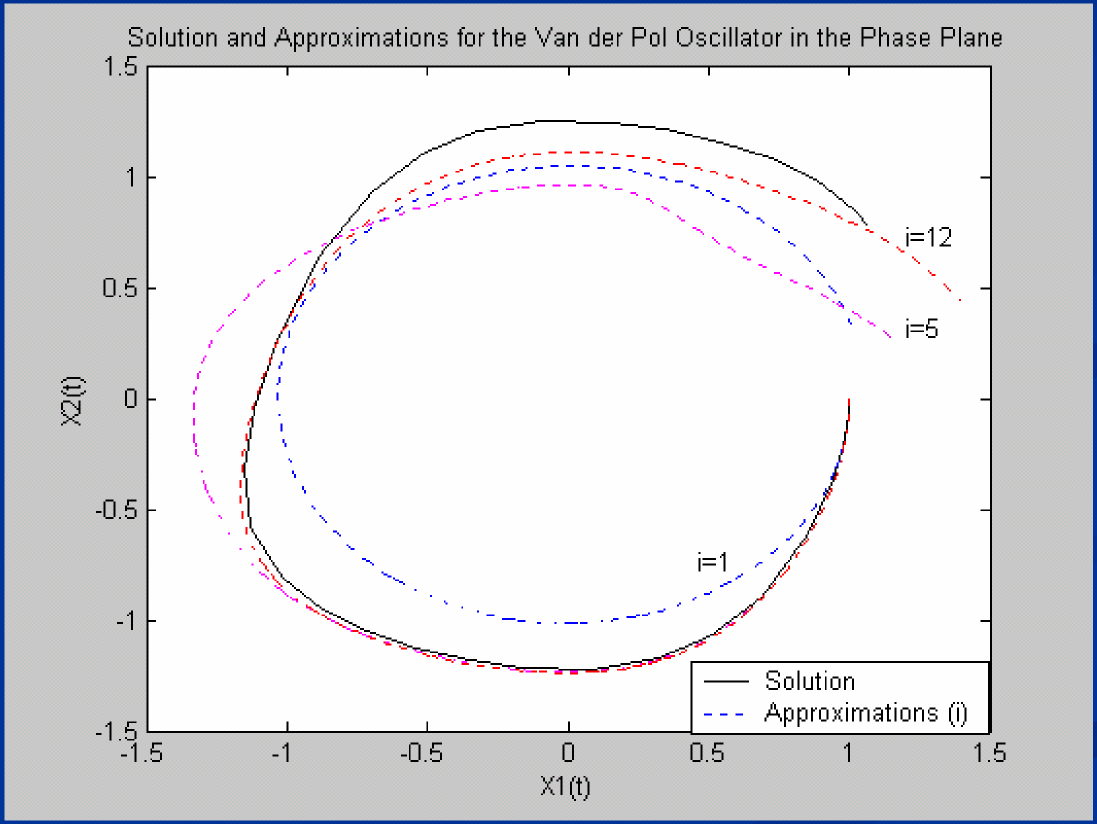


Fig 6. *Solution and Approximations of the Van der Pol Oscillator*

Fault Detection for Nonlinear Systems

Fault – Unacceptable change or failure of at least one parameter or system characteristic from its designed or normal operating conditions

Fault Detection - Determination of system faults and the time of the on-set faults

Model based-approach

(based on the mathematical model)

↓

State Estimation-Based Approach

↓

Reconstruct plant states and generate residuals by comparing estimated outputs with the measurements

Type of Fault: Abrupt and Additive

Fault Detection for LTV Systems

LTV SYSTEM:

$$\dot{x}(t) = Ax + B_u u + B_w w + F_1\mu_1 + F_2\mu_2$$

$$y(t) = Cx + v, x(t_0) = x_0$$

Where u control input — v sensor noise

y measurement — μ_1, μ_2 faults (functions of time)

w process noise — F_1, F_2 faults directions

Find linear observer and residual

$$\dot{\hat{x}}(t) = A\hat{x} + B_u u + L(y - C\hat{x})$$

$$r = \hat{H}(y - C\hat{x})$$

Objective → To find a residual primarly affected by the target fault and minimally by the nuisance faults

LTV Designs

- Chen,R., Mingori,D. and Speyer, J. (2003). Optimal stochastic fault detection filter, *Automatica*, 39, 377-390.
- Xu,A., Zhang, Q. (2002). Fault Detection and Isolation based on Adaptive Observers for Linear Time Varying Systems , *15th Triennial World Congress*, Barcelona, Spain.
- Edelmayer,A., Bokor,J., Szigeti,F. and Keviczky, L. (1997). Robust Detection Filter in the Presence of Time-Varying System Perturbations, *Automatica*, **33, No. 3**, 471-475.

Design proposed: "Optimal stochastic fault detection filter" (Chen, Mingori and Speyer)

- Algorithm is easy to program
- Extends the result of the UIO to the time varying case

Steps in design:

1. Find the filter gain by solving a Ricatti equation

$$\dot{P} = AP + PA^T - PC^T V^{-1} CP + \frac{1}{\gamma} F_2 Q_2 F_2^T - F_1 Q_1 F_1^T + B_w Q_w B_w^T, P(t_0) = P_0$$

$$L = PC^T V^{-1}$$

2. Find the projector

$$\hat{H} = [\rho_m \rho_{m-1} \ldots \rho_{p_2+1}][\rho_m \rho_{m-1} \ldots \rho_{p_2+1}]^T$$

where $CPC^T \rho = \rho\lambda, \quad p_2 = \dim F_2, \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$

3. Calculate the estimate

$$\dot{\hat{x}}(t) = A\hat{x} + B_u u + L(y - C\hat{x})$$

4. Obtain the residual $r = \hat{H}(y - C\hat{x})$

- Alcorta Garcia,E. and Frank, PM. (1997). Deterministic nonlinear observer-based approaches to fault diagnosis: a survey, *Control Engineering Practice,* **5**, 663-670.

- Yang, H. and Saif, SM. (1996) Monitoring and diagnostics of a class of nonlinear system using a nonlinear unknown input observer, *Proceedings of the 1996 IEEE International Conference on Control Applications,* Dearborn, MI, September 15-18. (Decomposition of the nonlinear system)

- Frank,P.M., (1994). On-line fault detection in uncertain nonlinear systems using diagnostic observers: a survey,*Int. J. Systems Sci.* **25,** 2129-2154.

- Seliger, R. and Frank, P.M. (1991) Fault-Diagnosis by disturbance decoupled nonlinear observers, *Proceedings of the 30th IEEE Conference on Decision and Control,* Brighton, England, 2248-2253. (Nonlinear state transformations)

Fault Detection for Nonlinear Systems

Given a nonlinear system

$$\dot{x}(t) = A(x)x(t) + B(x)u(t) + F_1(x)\mu_1(t) + F_2(x)\mu_2(t)$$
$$y(t) = C(x)x(t), x(t_0) = x_0$$

1. Reduction to a sequence of linear time varying approximations

$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t) + B(x^{[i-1]}(t))u^{[i]}(t) + F_1(x^{[i-1]}(t))\mu_1(t) + F_2(x^{[i-1]}(t))\mu_2(t)$$
$$y^{[i]}(t) = C(x^{[i-1]}(t))x^{[i]}(t), x^{[i]}(0) = x_0$$

2. Design of the Fault Detection Filter at each TV approximation

$$\dot{x}^{[i]}(t) = A(\hat{x}^{[i-1]}(t))\hat{x}^{[i]}(t) + B(x^{[i-1]}(t))u^{[i]}(t) + L^{[i]}(t)(y^{[i]}(t) - C(\hat{x}^{[i-1]}(t)\hat{x}^{[i]}(t))$$

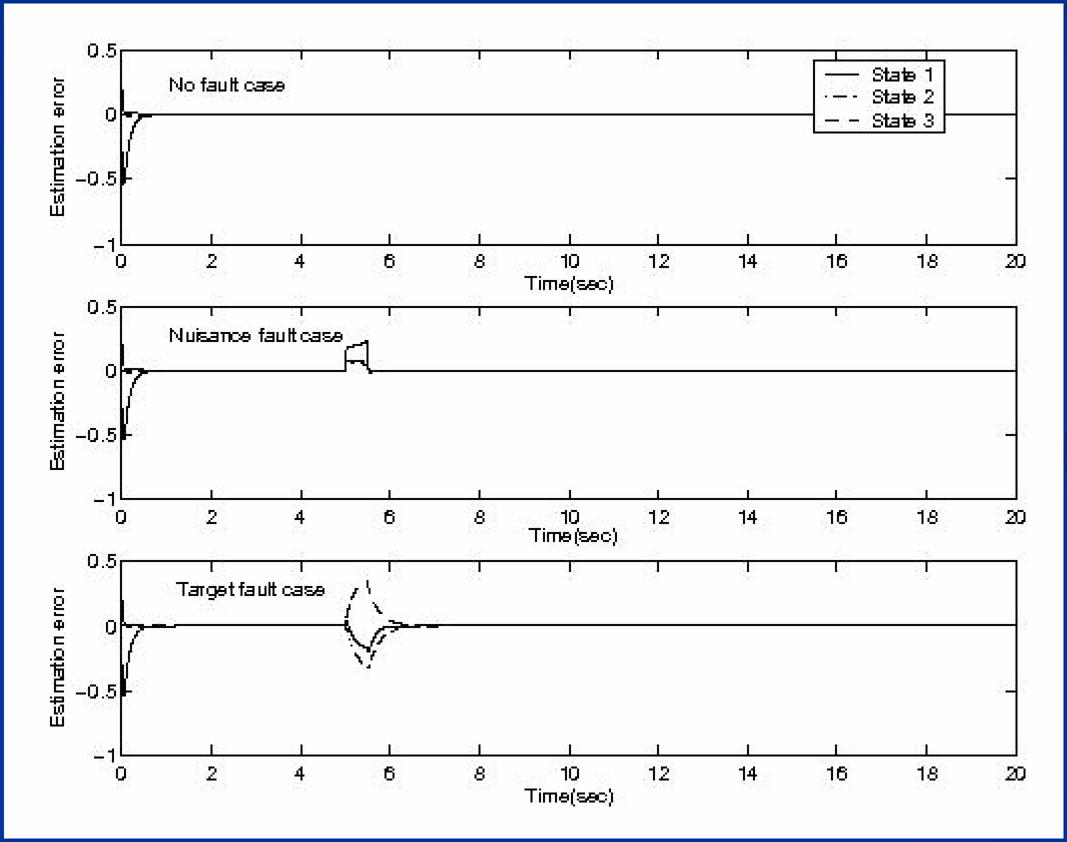
$$r^{[i]}(t) = \hat{H}^{[i]}(t)(y^{[i]}(t) - C(x^{[i-1]}(t))\hat{x}^{[i]}(t))$$

3. Obtain the residual at the final approximation

Nonlinear Example

$$\dot{x}(t) = \begin{pmatrix} -8 & x_2 & 0 \\ x_3 & -5 & x_1 \\ -x_2 & x_3 & -x_2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mu_1(t) + \begin{pmatrix} 5-2\cos(x_1) \\ 1 \\ 1+\sin(x_2) \end{pmatrix} \mu_2(t)$$

$$y(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(t)$$



$\mu_1 = \mu_2$

impulses of magnitude 3 at t=5 to t=5.5 sec.

I.C.

$x(0) = [.2,.2,.4]$

$\hat{x}(0) = [.5,0,.2]$

Fig 8. *Estimation error*

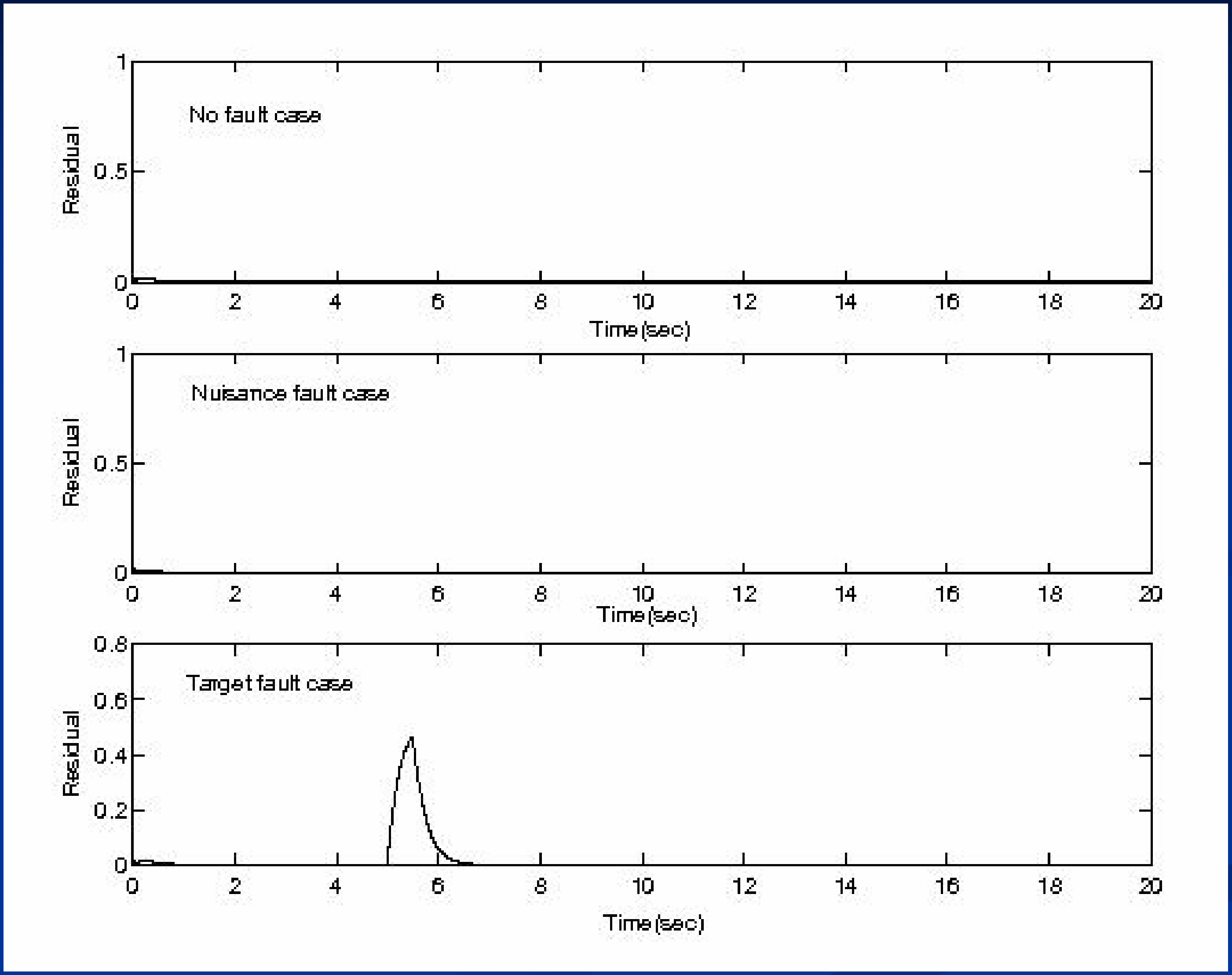


Fig 9. *Time response of the residual at different fault cases (gamma=10e-4)*

Conclusions

- The residual is primarily affected by the target fault and minimally by the nuisance faults
- New method to design a fault detection filter for nonlinear systems
- Sometimes there are problems with the solution of the Ricatti equation

Future Work

- Study of the required number of approximations
- Combine fault detection with control
- Reconstruction of the fault
- Further study of the limiting case

General Conclusions and Recommendations

- Present two different linearization techniques and their application to different problems in Control Theory
- The choice of the linearization technique depends on the specific problem to be solved
- Important to know ways of attacking nonlinear problems
- Further study of the previous linearization techniques and applications
- Study and comparison of the techniques

Thank you