On Complex Dynamical Networks

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Complex Networks:

Some Typical Examples
Complex Network Example: Internet

(William R. Cheswick)
Complex Network Example: WWW

(K. C. Claffy)
Complex Network Example: HTTP

(Bradley Huffaker)
Complex Network Example: Telecomm Networks

(Stephen G. Eick)
Complex Network Example: Routes of Airlines
Complex Network Example: Usenet

(Naveen Jamal)
Complex Network Example: VLSI Circuits, CNN
Complex Network Example: Biological Networks
Complex Network Example: Arts 😊
Complex Networks: Topics for Today

- Network Topology
- Three Approaches:
  - Random Graphs
  - Small-World Networks
  - Scale-Free Networks
- Network Control
- Network Synchronization
A **network** is a set of **nodes** interconnected via **links**

**Examples:**
- Internet: **Nodes** – routers  **Links** – optical fibers
- WWW: **Nodes** – document files  **Links** – hyperlinks
- Scientific Citation Network: **Nodes** – papers  **Links** – citation
- Social Networks: **Nodes** – individuals  **Links** – relations

**Nodes** and **Links** can be anything depending on the context
Network Topology

- Complex networks have been studied via Graph Theory - Erdös and Rényi (1960) – ER Random Graphs

- ER Random Graph model dominates for 40 some years

- ....... till today .......

- Availability of huge databases and supper-computing power have led to a rethinking of approach ...

- Two significant recent discoveries are:
  - **Small-World** effect (Watts and Strogatz, Nature, 1998)
  - **Scale-Free** feature (Barabási and Albert, Science, 1999)
ER Random Graph Models

Features:

- **Connectivity:** Poisson distribution
- **Homogeneous nature:** each node has roughly the same number of links

Erdős-Rényi


N nodes, each pair of node is connected with probability p

\[ p_{ER} = 0 \]

\[ p_{ER} = 0.2 \]
Small-World Networks

Features:
(Similar to ER Random Graphs)

- **Connectivity distribution**: uniform but decays exponentially
- **Homogeneous nature**: each node has roughly the same number of links

### Watts-Strogatz
(Nature 393, 440 (1998))

N nodes forms a regular lattice. With probability p, each edge is rewired randomly.

\[ p_{ws} = 0 \]

\[ p_{ws} = 0.3 \]
Scale-Free Networks

Features:

- **Connectivity:** in power-law form
- **Non-homogeneous nature:**
  a few nodes have many links but most nodes have very few links

(Hawoong Jeong)
Some Basic Concepts

- (Average) Distance
- Clustering Coefficient
- Degree and Degree Distribution
Average Distance

- **Distance** $d(n,m)$ between two nodes $n$ and $m$ = the number of links along the shortest path connecting them
- **Diameter** $D = \max\{d(n,m)\}$
- **Average distance** $L = \text{average over all } d(n,m)$

- Most large and complex networks have small $L$ $\Rightarrow$ small-world feature
Clustering Coefficient

- **Clustering Coefficient** \( C \) of a network:
  - \( 0 < C < 1 \)
  - \( C = 1 \) iff every pair of nodes are connected
  - \( C = 0 \) iff all nodes are isolated

- Most large and complex networks have large \( C \) \( \Rightarrow \) small-world feature
Degree and Degree Distribution

- **Degree** $k(n)$ of node $n = \text{total number of its links}$
- **The spread of node degrees over a network is characterized by a distribution function:**
  
  $P(k) = \text{probability that a randomly selected node has exactly } k \text{ links}$
Degree Distribution

- Completely regular lattice:
  \[ P(k) \sim \text{Delta distribution} \]
- Most networks: \( P(k) \sim k^{\{-\gamma\}} \) (power law)
  - scale-free feature
- Completely random networks:
  \[ P(k) \sim \text{Poisson distribution} \]
- (Regular) delta \( \leftrightarrow \) \( k^{\{-\gamma\}} \leftrightarrow \text{Poisson (Random)} \)
## Comparison

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<th>E-R random graph model</th>
<th>real-life complex networks</th>
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<td><strong>Ave. Distance</strong></td>
<td>Small / Large</td>
<td>Small</td>
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<tr>
<td><strong>Clustering</strong></td>
<td>Small</td>
<td>Large (Small-world feature)</td>
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<td><strong>Degree Distribution</strong></td>
<td>Binomial / Poisson</td>
<td>Power-law (Scale-free feature)</td>
</tr>
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Three Typical Examples

- World Wide Web
- Internet
- Scientific Collaboration Network

(Bradley Huffaker)
1. WWW

World Wide Web


Nodes: WWW documents

Links: URL links

800 million documents
(S. Lawrence, 1999)

ROBOT: collects all
URL’s found in a
document and follows
them recursively
1. World Wide Web

- **Average distance**
  - Computed: Average distance $L = 19.0$
  - Diameter $L = 19 \rightarrow$ at most 19 clicks to get anywhere

- **Degree distribution**
  - Outgoing edges: $P_1(k) \sim k^{\gamma_1}$
    - $\gamma_1 = 2.38 \sim 2.72$
  - Incoming edges: $P_2(k) \sim k^{\gamma_2}$
    - $\gamma_2 = 2.1$
2. Internet

(Computed in 1995-1999, at both domain level and router level)

- **Average distance**
  - \( L = 4.0 \)
  - ER Random Graph model: \( L = 10 \) (too large)
  - So, Internet is a small-world network

- **Degree distribution**
  - Obey power law: \( P(k) \sim k^{-\gamma}, \gamma = 2.2 \sim 2.48 \)
  - So, Internet is a scale-free network

- **Clustering coefficient**
  - \( C = 0.3 \)
  - ER Random Graph model: \( C = 0.001 \) (too small)

→ Small-world network is a better model for the Internet
3. Scientific Collaboration Network

- **Pál Erdős (1913-1996)**
- Oliver Sacks: "A mathematical genius of the first order, Paul Erdős was totally obsessed with his subject - he thought and wrote mathematics for nineteen hours a day until the day he died. He traveled constantly, living out of a plastic bag, and had no interest in food, sex, companionship, art - all that is usually indispensable to a human life."

-- The Man Who Loved Only Numbers (Paul Hoffman, 1998)
3. Scientific Collaboration Network

- **Erdős Number:**
  - Erdős published > 1,600 papers with > 500 coauthors in his life time
  - Published 2 papers per month from 20-year old to die of age 83
  - Main contributions in modern mathematics: Ramsey theory, graph theory, Diophantine analysis, additive number theory and prime number theory, ...

- **My Erdős Number is 2:**
  - P. Erdős – C. K. Chui – G. R. Chen

- Erdős had a (scale-free) small-world network of mathematical research collaboration
3. Scientific Collaboration Networks

- Databases of Scientific Articles - showing coauthors:
  - Los Alamos e-Print Archives: preprints (1992 - )
  - Medline: biomedical research articles (1961 - )
  - Stanford Public Information Retrieval System (SPIRES): high-energy physics articles (1974 - )
  - Network Computer Science Technical Reference Library (NCSTRL): computer science articles (10 years records)

- Computed for 10,000 to 2 million nodes (articles) over a few years → They are all small-world and scale-free (with power-law degree distributions)

Small-World Network Example: Language

- Words in human language interact like a small-world network
- Human brain can memorize about $10^4 \sim 10^5$ words (Romaine, 1992)
- Average distance between two words $d = 2 \sim 3$
- Degree distribution obeys a scale-free power-law: $P(k) = k^{\gamma-1}$, $\gamma = 3$

(Cancho and Sole)

FIG. 1. A possible pattern of wiring in $\Omega_z$. Black nodes are common words and white nodes are rare words. Two words are linked if they co-occur significantly.
**Actor connectivities**

- **nodes**: actors
- **edges**: casted jointly

Days of Thunder (1990)
Far and Away (1992)
Eyes Wide Shut (1999)

\[ N = 212,250 \text{ actors} \]
\[ \langle k \rangle = 28.78 \]

\[ P(k) \sim k^{-\gamma} \]
\[ \gamma = 2.3 \]

**Electrical powergrid**

- **nodes**: generators, transformers
- **edges**: transmission lines

The electrical power grid of the western United States

\[ \gamma \approx 4 \]
**SCIENCE CITATION INDEX**

**Nodes:** papers  
**Links:** citations

\[ P(k) \sim k^{-\gamma} \]

\[ \gamma = 3 \]

**INTERNET BACKBONE**  
(S. Redner, 1998)

\[ P(k=500) \sim 10^{-6} \]

\[ N_{WWW} \sim 10^9 \]

\[ \Rightarrow N(k=500) \sim 10^3 \]

**METABOLIC NETWORK**

**Nodes:** chemicals (proteins, substrates)  
**Links:** chemical reactions

\[ P(k) \sim k^{-2.2} \]

\[ \langle k_{in} \rangle = 6.5 \]

\[ \langle k_{out} \rangle = 3.4 \]
All large networks for which topological information is available follow

\[ P(k) \sim k^{-\gamma} \]

<table>
<thead>
<tr>
<th>WWW (in)</th>
<th>WWW (out)</th>
<th>Actor</th>
<th>Citation index</th>
<th>Power-grid</th>
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<tbody>
<tr>
<td>( \gamma = 2.1 )</td>
<td>( \gamma = 2.45 )</td>
<td>( \gamma = 2.3 )</td>
<td>( \gamma = 3 )</td>
<td>( \gamma = 4 )</td>
</tr>
</tbody>
</table>

All network models predict

\[ P(k) \sim e^{-k} \]

**SUMMARY**

All large networks for which topological information is available are scale free, and follow

\[ P(k) \sim k^{-\gamma} \]

**Origin:**
- growing nature of the real networks
- preferential attachment

**Model:** \( P(k) \sim k^{-\gamma} \)

**Mean field theory:** \( \gamma = 3, \quad k(t) \sim t^{1/2} \)

**UNIVERSALITY?**

www.nd.edu/~alb
A model for scale-free network generation

W. Aiello, F. Chung and L. Y. Lu (2001)

- Start with no nodes and no links
- At each time, a new node is added with probability $p$
- With probability $q$, a random edge is added to the existing nodes
- Here, $p + q = 1$

**Theorem:** The degree distribution of the network so generated satisfies a power law with $\gamma = 1 + 1/q$

- For $q = 1$, $\gamma = 2$; for $q = \frac{1}{2}$, $\gamma = 3$; hence, $2 < \gamma < 3$
Controlling Complex Dynamical Networks

- **De-coupling control:**
  
  Make use of coupling / de-coupling

- **Pinning control:**
  
  Only pinning a small fraction of nodes

**Random / Specific pinning:**

- **R:** Pin a fraction of randomly selected nodes
- **S:** First pin the most important node
  Then select and pin the next important node
  Continue ... till control goal is achieved
Pinning Control Example

A network with 10-nodes generated by the B-A scale-free model ($N=10, m=m_0=3$)
X. Li and X. F. Wang, APWCCS (2003)
Network Synchronization

- Network Synchronization
  - Complex Dynamics

- Network Synchronization
  - Synchronization in Small-World Networks
  - Synchronization in Scale-Free Networks
Synchronization in Globally Connected Networks

Observation:

No matter how large the network is, a global coupled network will synchronize if its coupling strength is sufficiently strong.

Good – if synchronization is useful
Synchronization in Locally Connected Networks

Observation:
No matter how strong the coupling strength is, a locally coupled network will not synchronize if its size is sufficiently large.

Good - if synchronization is harmful
Synchronization in Small-World Networks

Start from a nearest-neighbor coupled network

Add a link, with probability $p$, between a pair of nodes

Good news: A small-world network is easy to synchronize!

Synchronization in Scale-Free Networks

- **Robust** against random attacks and random failures

- **Fragile** to intentional attacks and purposeful removals of ‘big’ nodes

Both are due to the extremely inhomogeneous connectivity distribution of scale-free networks

SCI papers: Complex Networks

![Bar chart showing the number of SCI papers on Complex Networks from 1998 to 2004. The number of papers increases significantly from 1998 to 2004.](chart.png)
EI papers: Complex Networks
SCI papers: Small-World Networks
EI papers: Small-World Networks

![Graph showing the number of EI papers on Small-World Networks from 1998 to 2004. The number of papers increases from 1998 to 2004, peaking at 211 in 2004.]
SCI papers: Scale-Free Networks
EI papers: Scale-Free Networks
So much for today …
Main References

- **Overview Articles**

- **Some Related Technical Papers**
  - [http:www.ee.cityu.edu.hk/~gchen/Internet.htm](http:www.ee.cityu.edu.hk/~gchen/Internet.htm)
Thank You!