Global Theory of Nonlinear Dynamical Systems

Steve Banks
Department of Automatic Control and Systems Engineering University of Sheffield
- All integrable Hamiltonian systems have constant energy surfaces which are n-dimensional toruses or products of Klein bottles.
Differentiable Manifolds
A dynamical system on $M$ is a map

$$\sigma : \mathbb{R} \times M \rightarrow M$$

such that

$$\sigma(0,x)=x \text{ for all } x \text{ in } M$$

and

$$\sigma(t,\sigma(s,x))=\sigma(t+s,x) \text{ for all } s,t \text{ in } \mathbb{R}, x \text{ in } M$$
To relate a dynamical system to a differential equation, define the tangent vector $X$ at $x$ by

$$X(f) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ f(\sigma(\varepsilon, x)) - f(x) \right]$$

In terms of components:

$$\frac{d}{dt} \sigma_i(t, x) = X_i(\sigma(t, x))$$
A subset $C \subseteq M$ is invariant for the dynamical system $\sigma$ if

$$\sigma(t, C) \subseteq C$$

Examples:

(1) Any orbit (solution trajectory)
(2) An equilibrium point
(3) A periodic orbit
(4) A chaotic attractor
Finding the general topological structure of nonlinear dynamical systems is very difficult – index theories give some insight into the global structure.

The index at P is the total change of angle around the loop.
Here are the indices of the equilibrium points of simple 2-dimensional linear systems:

<table>
<thead>
<tr>
<th>Node (stable or unstable)</th>
<th>centre</th>
<th>saddle</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>+1</td>
<td>-1</td>
</tr>
</tbody>
</table>

The index is a (topological) **invariant** – i.e. it does not matter what the exact shape is.
For a general nonlinear system, the equilibria are of the form:

\[ \text{Index} = 1 + \frac{(e-h)}{2} \]

where \( e \) = number of elliptic sectors

\( h \) = number of hyperbolic sectors
The sum of the indices of the critical points is always the Euler characteristic of the surface (i.e. \(2(1-p)\)) where \(p\) is the genus of the surface.
Morse index

Conley Index

Maslov index

The Poincare theorem generalises to higher dimensions by the use of algebraic topology.
Planar, 2-dimensional systems cannot contain knots.
Consider the Lorenz system:

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1) \\
\dot{x}_2 &= x_1(r - x_3) - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3
\end{align*}
\]
Singular line
Knots cannot exist on two-dimensional Euclidean space, but they can occur in dynamical systems on general surfaces. For example, here are two torus kots:
Now consider the question – how many topologically distinct knots can there be on a surface of genus $g$ and what does the rest of the dynamics look like?

**Theorem** On a surface of genus $g$ there can exist at most $g$ topologically distinct knots. Then the dynamics modulo the knots is equivalent to the dynamics of a system defined on a sphere with at least $2g$ equilibria of index 1.

For the proof we cut the surface along the knots and shrink them to zero. Each knot is a cycle of index 1. The resulting object is topologically a sphere.
Example
Consider first the case of a torus.
System with a single pole.

\[ \dot{z} = z^{-k} \]

It can be shown that this system has index \(-k\) at the pole.

Similarly, the system with a single zero

\[ \dot{z} = z^k, \quad k > 1 \]

Has index \(+k\) at the zero.
In order to generate general systems on the torus, we can use the theory of elliptic functions.

**Definition** An elliptic function is a meromorphic function from the torus to the Riemann sphere, which is doubly periodic on a lattice

\[ \Omega = \{m\omega_1 + n\omega_2 : m, n \text{ integers, } \omega_1, \omega_2 \text{ complex} \} \]

For example we have the Weirestrass function

\[ \wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right) \]
and the Weirestrass $\sigma$ function

$$\sigma(z) = z \cdot \prod_{\omega \in \Omega}' \left\{ \left(1 - \frac{z}{\omega} \right) \exp \left( \frac{z}{\omega} + \frac{1}{2} \left( \frac{z}{\omega} \right)^2 \right) \right\}$$

The importance of this function is that any elliptic function with given poles and zeros can be made from $\sigma$ functions:

$$f(z) = \frac{\prod_{i=1}^{r} \prod_{j=1}^{k_i} \left( \sigma(z - a_i) \right)^j}{\prod_{i=1}^{s} \prod_{j=1}^{\ell_i} \left( \sigma(z - b_i) \right)^j}$$
A general meromorphic system on a torus is therefore given by

\[ \dot{z} = E(z) \]

where \( E \) is an elliptic function.

**Theorem** Suppose that we choose distinct points \( \{p_1, \ldots, p_K\} \) on a torus such that we may find a set of points \( \{q_1, \ldots, q_K\} \) in the fundamental parallelogram for which

\[ \sum_{i=1}^{k_1} q_i + \sum_{i=k_1+1}^{k_2} \left( \frac{e_i}{2} + 1 \right) q_i \sim \sum_{k_2+1}^{K} \left( \frac{h_i}{2} - 1 \right) q_i \pmod{\Omega} \]

then there exists a dynamical system on the torus for which
\[ \begin{cases} p_1, \ldots, p_{k_1} \text{ are each surrounded by } 1 \text{ parabolic sector} \\
p_i \text{ is surrounded by } e_i \text{ elliptic sectors, for } k_1 + 1 \leq i \leq k_2 \\
p_i \text{ is surrounded by } h_i \text{ hyperbolic sectors, for } k_2 + 1 \leq i \leq K. \end{cases} \]

**Examples**

The simplest, nontrivial example is

\[ \dot{z} = \wp(z) \]

The Weirestrass P function has one zero and one pole in the fundamental region.
Phase portrait of $\frac{dz}{dt} = \text{WeierstrassP}(z)$

$x = \text{Re}(z)$

$y = \text{Im}(z)$
Now consider the system given by

\[ \dot{z} = \frac{\sigma(z - 0.25 - 0.25i)\sigma(z - 0.75 - 0.75i)}{(\sigma(z - 0.5 - 0.5i))^2} \]

(two simple zeros and a pole of order 2).
This case is much more difficult since we cannot tesselate the Euclidean plane with octagons, dodecagons, etc. To do this we need to use hyperbolic geometry.

- Euclidean geometry, the fifth postulate, i.e. through any point there is precisely one line parallel to a given line not containing the point.

- Lobachevsky and Gauss showed that the fifth postulate is independent of the rest.

- Spaces of negative curvature have hyperbolic geometry
We can use the upper half-space model.

Periodic functions and fundamental parallelograms in the case of tori become automorphic functions and fundamental regions in hyperbolic space with $4g$ sides.
Global theory of nonlinear systems is important in understanding the overall behaviour of systems.

It is related strongly to the topology of the underlying manifold of the system.

We can find all meromorphic systems on surfaces of genus $p$.

In dimension 3 it is much more difficult!