CDMA Design Through Asymptotic Analysis: Fading Channels

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Abstract

Using asymptotic analysis, we show how flat fading affects CDMA with linear receivers. Specifically, we let the number of users grow without bound while the ratio of number of users to spreading sequence length is kept fixed to a value α. We treat separately the case of flat, slow fading with lognormal shadowing (nonergodic channel) and of flat, fast Rayleigh fading (ergodic channel). For the former channel we derive the outage probability, while for the latter we compute the channel capacity. We show in particular that, with fading, as $\alpha \to \infty$ the system capacity tends to the same limit of 1.44 bit/symbol as for the non-fading channel. On the contrary, the outage probability exibits a floor for all α values when a single-user matched filter is employed, while with a linear minimum-mean-square-error receiver the floor is present only for $\alpha > 1$.

1 Introduction

We examine a CDMA system with error-control coding operating on a channel affected by flat fading. We treat separately the case of flat, slow fading with lognormal shadowing (nonergodic channel) and of flat, fast Rayleigh fading (ergodic channel). For the former channel we derive the outage probability, while for the latter we compute the channel capacity. The receiver consists of a linear front-end, viz., either a single-user matched filter (SUMF) detector or a linear minimum-mean-square error receiver (MMSE) detector [6], followed by a single-user decoder. The key performance measure here is the Signal-to-Interference plus Noise Ratio (SINR) at the output of the linear filter: users' quality of service can be expressed in terms of a target SINR. The SINR cumulative distribution function (cdf) yields immediately the *outage probability*, i.e., the probability that the actual SINR, say β, is below the required SINR target, denoted β_0. In general, β_0 depends on the coding scheme. Our study is asymptotic, in the sense that the number of users grows without bound, while the ratio of number of users to spreading-sequence length is kept fixed. In this work, after a description of the system model, we shall examine the outage probability of slow-fading channels in Section 3, and the capacity of fast-fading Rayleigh channel in Section 4.

2 System model

We consider a single-cell, chip-synchronous DS-CDMA system. Our model involves K users, random spreading sequences, and a spreading-sequence length $L = T/T_c$, where $1/T$ is the symbol rate and $1/T_c$ the chip rate. We assume (as in [3, 5]) a large number of users ($K \to \infty$) and $K/L \to \alpha$ (a constant "channel load" as the length of the spreading sequences increases to accommodate the K users). Let the empirical distribution function of the received interfering power from all users converge, as $K \to \infty$, to $F_{\mathcal{P}}(x)$. Specifically, denote by $\mathcal{P}_k$ the power received from user k, and by $F_K(x)$ the empirical cumulative distribution function of the received powers:

$$F_K(x) = \frac{1}{K}\sum_{k=1}^{K} \Im\{\mathcal{P}_k \leq x\}$$

where $\Im\{\mathcal{A}\}$ is the indicator function of the event $\mathcal{A}$. We require that $F_K(x) \to F_{\mathcal{P}}(x)$ almost everywhere as $K \to \infty$. Thus, $F_{\mathcal{P}}(x)$ is the average cumulative distribution function of the power received from a user picked at random (with uniform probability $1/K$) from the population of $K \to \infty$ users. In our previous paper [2] we chose for $F_{\mathcal{P}}(x)$ the step function

$$F_{\mathcal{P}}(x) = u(x - \mathcal{P}_0)$$

corresponding to a situation in which the power received from all users is $\mathcal{P}_0$. From [3, 4], with single-user matched filter (SUMF) reception and under mild convergence conditions we have the asymptotic SINR:

$$\beta = \frac{\mathcal{P}}{N_0 + \alpha \int_0^\infty x \, dF_{\mathcal{P}}(x)}, \tag{1}$$

where $\mathcal{P}$ is the useful power received from reference user, and N_0 is the variance of the additive white Gaussian noise samples. With linear minimum mean-square error (MMSE) reception the asymptotic SINR β is the unique real nonnegative solution of

$$\beta = \frac{\mathcal{P}}{N_0 + \alpha \displaystyle\int_0^\infty \frac{x\mathcal{P}}{\mathcal{P} + x\beta} \, dF_{\mathcal{P}}(x)}. \tag{2}$$

2.1 Methodological premises

The intricacies of dealing with fading channels are thoroughly described in [1], so we shall not delve into them here. Rather, we list a number of points that describe the rationale behind the calculations that follow. In particular, we motivate the different methods that are used to analyze the two situations of flat, slow fading and of flat, fast fading.

2.1.1 Flat, slow fading

Consider first flat, slow fading—not slow enough to be compensated by the power-control system, yet slow enough to make the channel nonergodic. The fading is due to lognormal shadowing. This situation occurs for example when the power control is imperfect, thus leaving a residual shadowing of 2 dB (say), rather than the 8 dB of the uncompensated shadowing. In this situation, (1) and (2) are still valid, provided that in lieu of $\mathcal{P}$ we write $z_1\mathcal{P}$, where z_1 is the lognormal RV modeling the shadowing in user-1 channel, and in lieu of $F_{\mathcal{P}}$ we write the limiting empirical distribution of the product $z_k\mathcal{P}_k$, where z_k is the lognormal RV modeling user-k channel and $\mathcal{P}_k$ is the power received from user k when there is no shadowing. Even if we choose all the $\mathcal{P}_k$ equal (to $\mathcal{Q}$, say), the distribution $F_{\mathcal{P}}$ is not a step function, but rather equals the distribution of the lognormal RV $z\mathcal{Q}$, where z is a generic lognormal RV. Moreover, since we have $z_1\mathcal{P}$ instead of $\mathcal{P}$, the value of β becomes a function of z_1, and hence a random variable. We will see in the following that β is proportional to z for both SUMF and MMSE. We may define an outage probability

$$P(\beta < \beta_0) \tag{3}$$

which will not be generally a step function as in the nonfaded channel [2], even in our limiting case of K and L both growing to infinity. The value of β_0 is chosen as in [2]. For example, if we use a nonideal code with rate R bit/symbol achieving the target performance at a certain E_b/N_0, we set $\beta_0 = RE_b/N_0$. If we consider instead an optimum code which operates at the Shannon limit for a Gaussian channel and we want a rate R, then we set

$$\beta_0 = 2^R - 1 \tag{4}$$

as we did in [2].

2.1.2 Flat, fast fading

This is the case of a fading so fast that it cannot be compensated by power control but constant over one symbol, and such as to make the channel ergodic. In this situation, (1) and (2) are still valid under the same modification as in the previous subsection, but now z_1 and z are distributed as the instantaneous power gain of the channel. Here we choose $\sqrt{z}$ and $\sqrt{z_1}$ to be Rayleigh-distributed, so that z_1 and z are exponentially distributed (with mean 1). Here the value of β (which, being a function of z_1, is a random variable) can be interpreted as the instantaneous SINR at the output of the linear receiver. With optimum (Gaussian) codes, the capacity is given by [1]

$$C = \mathbb{E}[\log_2(1+\beta)] \tag{5}$$

where the expectation $\mathbb{E}$ is taken with respect to the distribution of β. Eq. (5) can be evaluated by using numerical techniques. With SUMF, β is still exponentially distributed. With MMSE, we can expect the distribution of β to be highly complex, so that the average capacity will be even worse (there is no conceptual complication, though). A major departure from the situation of previous subsection occurs with the definition of outage probability: in fact the probability $P(\beta < \beta_0)$ has no practical significance anymore (although a widespread misconception leads it sometimes to be interpreted as an outage probability). This is due to the fact that β is the instantaneous SINR, and with coding and fast fading the probability of error does not depend on the instantaneous SINR (which is a RV), but rather on the statistics of SINR and on the code features, and is difficult to compute. We argue that in our case the sensible quantity is the *system capacity*, defined as

$$C_{\text{sys}} \triangleq \max \alpha C(\alpha) \quad \text{bit/s/Hz} \tag{6}$$

and we write $C(\alpha)$ rather than C to stress the fact that the capacity depends on α. If we want all users to transmit reliably at rate R, the maximum admissible α is

$$\alpha_{\max} \triangleq \max\{\alpha : R \leq C(\alpha)\} \tag{7}$$

In this way, we can still define a *system outage probability*: this is now a step function, taking on value 0 if $\alpha < \alpha_{\max}$, and 1 if $\alpha > \alpha_{\max}$.

3 Slow-fading channel

Let $\eta \triangleq \mathcal{P}/N_0 = RE_b/N_0$ be the signal-to-noise ratio for the single user.

3.1 SUMF receiver

With the SUMF receiver,

$$\beta = \frac{z_1 \mathcal{P}}{N_0 + \alpha \mathcal{P} \mathbb{E}[z]} = \frac{z_1 \mathcal{P}}{N_0 + \alpha \mathcal{P}}$$

where z_1 and z denote the power gains of the useful and interfering users, respectively, and follow a log-normal distribution with log-standard deviation σ dB and mean value $\mathbb{E}[z] = 1$. In other words,

$$10 \log_{10} z \sim \mathcal{N}(-\mu_z, \sigma)$$

where $\mu_z \triangleq \sigma^2 (\ln 10)/20$. Letting $\kappa_{\mathrm{s}}(\alpha, \eta) \triangleq 1/(\eta^{-1} + \alpha)$ the outage probability is given by

$$\begin{aligned} P_{\mathrm{out,s}} &= P(\beta \le \beta_0 = 2^R - 1) = P\left(z_1 \le (2^R - 1)/\kappa_{\mathrm{s}}(\alpha, \eta)\right) \\ &= 1 - Q\left[\sigma^{-1}\left(\mu_z + 10 \log_{10}[(2^R - 1)/\kappa_{\mathrm{s}}(\alpha, \eta)]\right)\right] \end{aligned} \tag{8}$$

where $Q(x) \triangleq P(\mathcal{N}(0, 1) > x)$.

3.2 MMSE receiver

With the MMSE receiver, the asymptotic SINR is the solution of the following equation, obtained from the substitutions $\mathcal{P} \to z_1 \mathcal{P}$ and $x \to z\mathcal{P}$ in (2):

$$\beta = \frac{z_1 \mathcal{P}}{N_0 + \alpha \mathcal{P} \mathbb{E}\left[\dfrac{z_1 z}{z_1 + \beta z}\right]} \tag{9}$$

where $\mathbb{E}[\cdot]$ denotes expectation with respect to z. This equation can also be expressed as follows:

$$z_1/\beta = \eta^{-1} + \alpha \mathbb{E}_z \left[\frac{z}{z_1/\beta + z}\right] (z_1/\beta) \tag{10}$$

and can be solved iteratively by the recursion

$$\begin{aligned} \xi_n &= \eta^{-1} + \alpha \mathbb{E}\left[\frac{z}{\xi_{n-1} + z}\right] \xi_{n-1} \\ &= \eta^{-1} + \frac{\alpha}{\sqrt{2\pi}} \xi_{n-1} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\xi_{n-1} 10^{(\mu_z - \sigma x)/10} + 1} \, dx \end{aligned}$$

initialized by $\xi_0 = 1$. This iteration converges to a constant $\lim_{n\to\infty} \xi_n = \kappa_{\mathrm{m}}(\alpha, \eta)$ as shown in the following subsection which is the solution ξ of eq. (10). Its solution can be written in the form

$$\beta = \kappa_{\mathrm{m}}(\alpha, \eta) z_1 \tag{11}$$

Note that β is proportional to z_1 by a deterministic factor depending only on the parameters α and η. The outage probability is

$$P_{\text{out,m}} = P\left(z_1 \leq (2^R - 1)/\kappa_{\text{m}}(\alpha, \eta)\right) = 1 - Q\left[\sigma^{-1}\left(\mu_z + 10\,\log_{10}[(2^R - 1)/\kappa_{\text{m}}(\alpha, \eta)\right)\right]$$

3.3 Outage probability floor

The above results show that with both receivers the outage probability may approach a nonzero limit as $\eta \to \infty$ (outage probability floor).

With the SUMF receiver, since $\lim_{\eta\to\infty} \kappa_{\text{s}}(\alpha, \eta) = \alpha^{-1}$, from (8) the outage probability floor is

$$P_{\text{out,s}} = 1 - Q\left[\sigma^{-1}\left(\mu_z + 10\,\log_{10}[(2^R - 1)\alpha]\right)\right] \tag{12}$$

for any value of α. This result derives from the fact that the SUMF receiver is not near-far resistant [6]: because of the fading, with nonzero probability some interferer is strong enoughto make the SINR fall below the outage threshold.

As for the MMSE receiver, we have an outage probability floor if and only if $\alpha > 1$, i.e., the number of users exceeds the spreading factor. In fact, letting $\eta \to \infty$ in (10) we obtain the following equation:

$$\alpha \mathbb{E}_z \left[\frac{\beta z}{z_1 + \beta z}\right] = 1$$

This equation has a solution if and only if $\alpha \geq 1$: in fact, the second factor in its LHS is always less than or equal to 1. This result is due to the lack of near-far resistance of the MMSE receiver when $\alpha > 1$ [6].

The outage probability floor is illustrated in Fig. 1, obtained by plotting P_{out} versus E_b/N_0 with the MMSE and SUMF receivers, $\alpha = 0.2, 0.5, 0.8, 1, 1.2, 1.5$, rates $R = 1$ and 2 bit/symbol, and $\sigma = 2$ and 8 dB. The outage probability degrades, as we see, by increasing either R, α, or σ, which represent the user rate, system load, and shadowing level, respectively.

4 Flat, fast fading

The channel capacity is given by

$$C(\alpha) = \mathbb{E}[\log_2(1 + \beta)]$$

Here, z and z_1 are exponentially-distributed RV's with unit mean. For the SUMF receiver, we get

$$C_{\text{s}}(\alpha) = \int_0^\infty \log_2\left(1 + \frac{z_1 \mathcal{P}}{N_0 + \alpha\mathcal{P}}\right) e^{-z_1} dz_1 = e^{1/\kappa_{\text{s}}(\alpha,\eta)}\,\text{Ei}(1, 1/\kappa_{\text{s}}(\alpha, \eta)) \log_2 e$$

For the MMSE receiver, equation (10) becomes

$$\begin{aligned} z_1/\beta &= \eta^{-1} + \alpha\,(z_1/\beta) \int_0^\infty \frac{z}{z_1/\beta + z} e^{-z}\,dz \\ &= \eta^{-1} + \alpha\,(z_1/\beta)[1 - (z_1/\beta)e^{z_1/\beta}\text{Ei}(1, z_1/\beta)] \end{aligned} \tag{13}$$

which yields $\beta = \kappa'_{\rm m}(\alpha,\eta) z_1$. Then, the capacity is given by

$$C_{\rm m}(\alpha) = \int_0^\infty \log_2[1+\kappa'_{\rm s}(\alpha,\eta) z_1] e^{-z_1} dz_1 = e^{1/\kappa'_{\rm m}(\alpha,\eta)} \,\mathrm{Ei}(1, 1/\kappa'_{\rm m}(\alpha,\eta)) \log_2 e \qquad (14)$$

Figure 5 shows the capacity curves for SUMF and MMSE as a function of $\mathcal{P}/N_0 = \eta$ for different values of α. Note that the capacity is always bounded for the SUMF receiver and for the MMSE receiver when $\alpha > 1$.

4.1 System capacity

We define the system capacity as

$$C_{\rm sys} = \max_{\alpha: R \leq C(\alpha)} \alpha\, C(\alpha)$$

Figure 3 shows both $C(\alpha)$ and $\alpha\, C(\alpha)$ versus α at $\eta = 6$, 10, 20 and 30 dB, respectively. They illustrate the following results.

- With the SUMF receiver, the product $\alpha\, C(\alpha)$ increases with α, while for the MMSE receiver it exhibits a maximum.

- As $\alpha \to \infty$, with both receivers $\alpha\, C(\alpha)$ approaches the same limit, 1.44 bit/symbol, regardless of η. In fact, with the SUMF receiver we have

  $$\lim_{\alpha\to\infty} \alpha\, C_{\rm s}(\alpha) = \lim_{\alpha\to\infty} \alpha\, e^{\eta^{-1}+\alpha}\, \mathrm{Ei}(1, \eta^{-1}+\alpha) \log_2 e = \log_2 e = 1.44 \text{ bit/symbol}$$

  With the MMSE receiver, when $\alpha \to \infty$, we have from (13):

  $$\xi = \eta^{-1} + \alpha\xi[1 - \xi\, e^{\xi}\, \mathrm{Ei}(1,\xi)] = \alpha\xi[1/\xi + O(1/\xi^2)] = \alpha[1 + O(1/\xi)]$$

  where we set $\xi = z_1/\beta$. Now, from $\xi = \alpha[1 + O(1/\xi)]$ we have $\xi = \alpha[1 + q/\xi]$ for some finite q when both ξ and α are sufficiently large. Thus, $\xi^2 - \alpha\xi - q = 0$, which can be solved as follows:

  $$\xi = \frac{1}{2}[\alpha + \sqrt{\alpha^2 + 4q}] = \frac{1}{2}[\alpha + \alpha\sqrt{1 + 4q/\alpha^2}] = \frac{1}{2}[\alpha + \alpha + 2q/\alpha + O(\alpha^{-3})]$$

  It follows that $\xi = \alpha + O(1/\alpha)$. Hence

  $$\lim_{\alpha\to\infty} \alpha\, C_{\rm m}(\alpha) = \lim_{\alpha\to\infty} \alpha\, e^{\xi}\, \mathrm{Ei}(1,\xi) \log_2 e = \lim_{\alpha\to\infty} \alpha[1/\xi + O(1/\xi^2)] \log_2 e = 1.44$$

  Since in a non-faded channel both SUMF and MMSE achieve the same limiting spectral efficiency of 1.44 bit/s/Hz [2], we have shown that fast Rayleigh fading does not reduce system capacity.

- The maximum system capacity with SUMF is achieved for $\alpha \to \infty$. On the contrary, with MMSE receivers an optimal $\alpha = \alpha_{\rm opt}$ exists which maximizes $\alpha\, C_{\rm m}(\alpha)$; thus, the system capacity is $\alpha_{\rm opt}\, C_{\rm m}(\alpha_{\rm opt})$. This optimum can be found by solving (13) for α:

  $$\alpha = \frac{\xi - \eta^{-1}}{\xi[1 - \xi\, e^{\xi}\, \mathrm{Ei}(1,\xi)]}$$

which yields

$$\alpha\, C_{\mathrm{m}}(\alpha) = \frac{\xi - \eta^{-1}}{\xi[1 - \xi\, e^{\xi}\, \mathrm{Ei}(1,\xi)]} e^{\xi}\, \mathrm{Ei}(1,\xi) \log_2 e$$

The optimum ξ can be obtained and Fig. 4 shows the system capacity and the optimal load versus the SNR η. The system capacity is increasing asymptotically linearly with the SNR. Note that α_{opt} has a minimum at SNR=30.9 dB (0.8635). This fact can be explained by noting that

1. For low SNR, the optimum linear MMSE receiver approaches the SUMF receiver since noise dominates MAI. In this case, system capacity is maximum when $\alpha \to \infty$, as with the SUMF receiver (additionally, the SNR$\to 0$).
2. For high SNR, the optimum linear MMSE receiver approaches the decorrelating receiver which is the optimal linear receiver in the absence of noise. In this case, system capacity is maximum for $\alpha \to 1$. In fact, if SNR$= \eta \to \infty$,

$$\alpha C_{\mathrm{m}}(\alpha) = e^{\xi} \frac{\mathrm{Ei}(1,\xi)}{1 - \xi e^{\xi} \mathrm{Ei}(1,\xi)} \log_2 e$$

   The rhs is a monotonically decreasing function of ξ so that the maximum is attained when $\xi = 0$ and

$$\lim_{\xi \to 0} \alpha = \lim_{\xi \to 0} \frac{1}{1 - \xi e^{\xi} \mathrm{Ei}(1,\xi)} = 1$$

Then, the continuity of $\alpha_{\mathrm{opt}}(\eta)$ implies that a maximum (≤ 1) exists.

4.1.1 Gaussian channel

For comparison, let us consider the channel capacity in the case of a purely Gaussian channel without fading where all users transmit with the same power $\mathcal{P}$ [3, 5]

$$C(\alpha) = \log_2(1 + \beta)$$

For the SUMF receiver the SINR is simply

$$\beta = \frac{1}{\eta^{-1} + \alpha}$$

For the MMSE receiver the SINR can be evaluated from

$$\beta = \frac{1}{\eta^{-1} + \frac{\alpha}{1+\beta}}$$

which yields

$$\beta = \frac{1}{2}\left[\eta(1-\alpha) - 1 + \sqrt{1 + 2\eta(1+\alpha) + \eta^2(1-\alpha)^2}\right]$$

Figure 5 show $C(\alpha)$ and $\alpha C(\alpha)$ for the two receivers for different $\mathcal{P}/N_0$. We observe that the Gaussian channel capacity is slightly higher than that of the fading channel, but for $\alpha \to \infty$ both converge to the same limit of 1.44 bit/symbol.

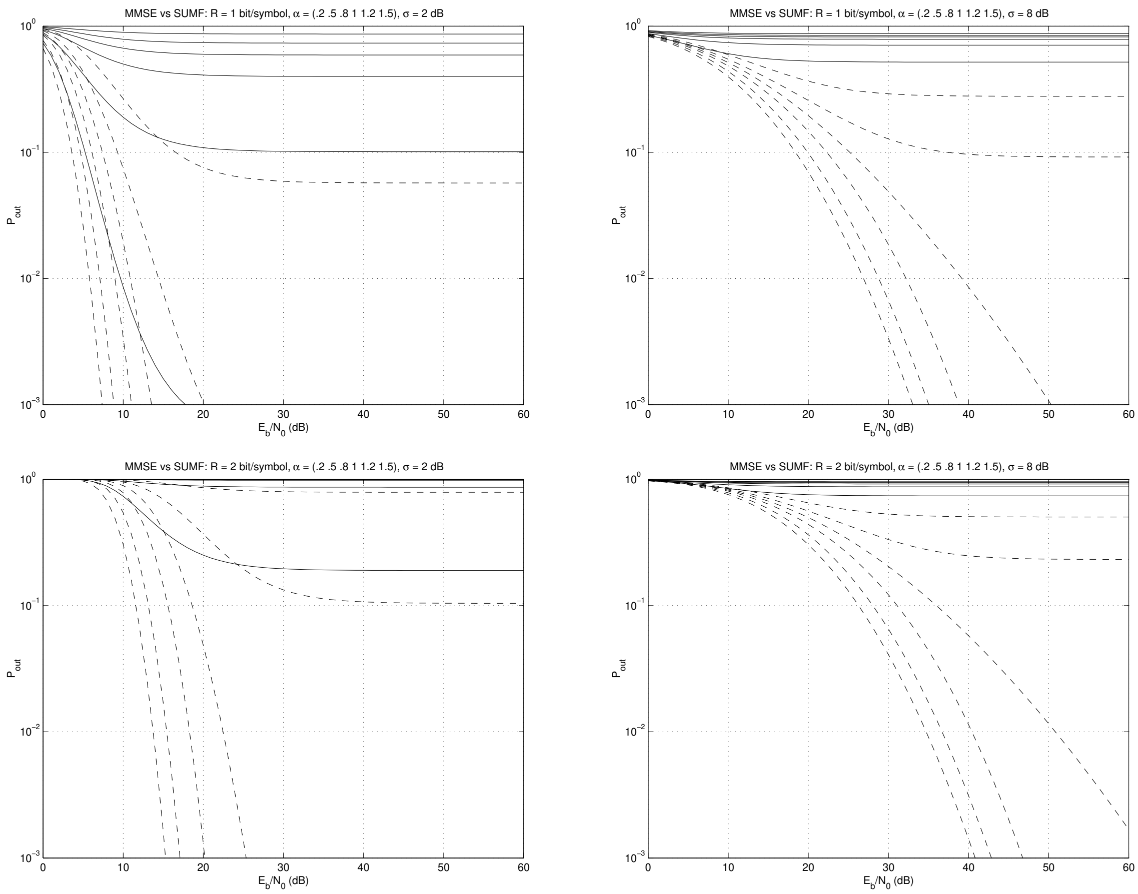

Figure 1: Outage probability with MMSE (dashed) and SUMF (solid) for rates $R = 1$ and 2 bit/symbol, $\sigma = 2$ and 8 dB, and $\alpha = 0.2, 0.5, 0.8, 1, 1.2, 1.5$.

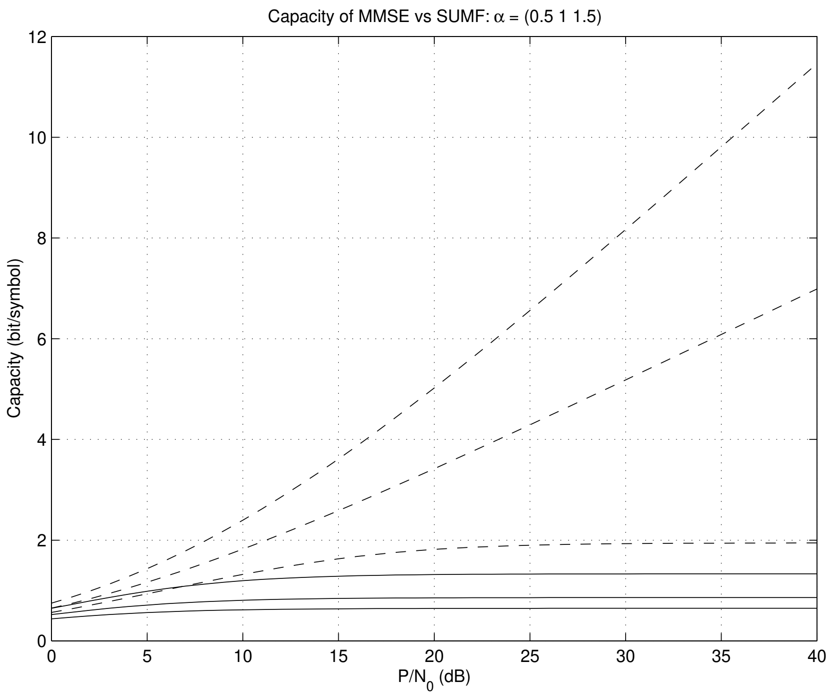

Figure 2: Capacity of MMSE (dashed) and SUMF (solid) (bit/symbol), $\alpha = 0.5, 1, 1.5$.

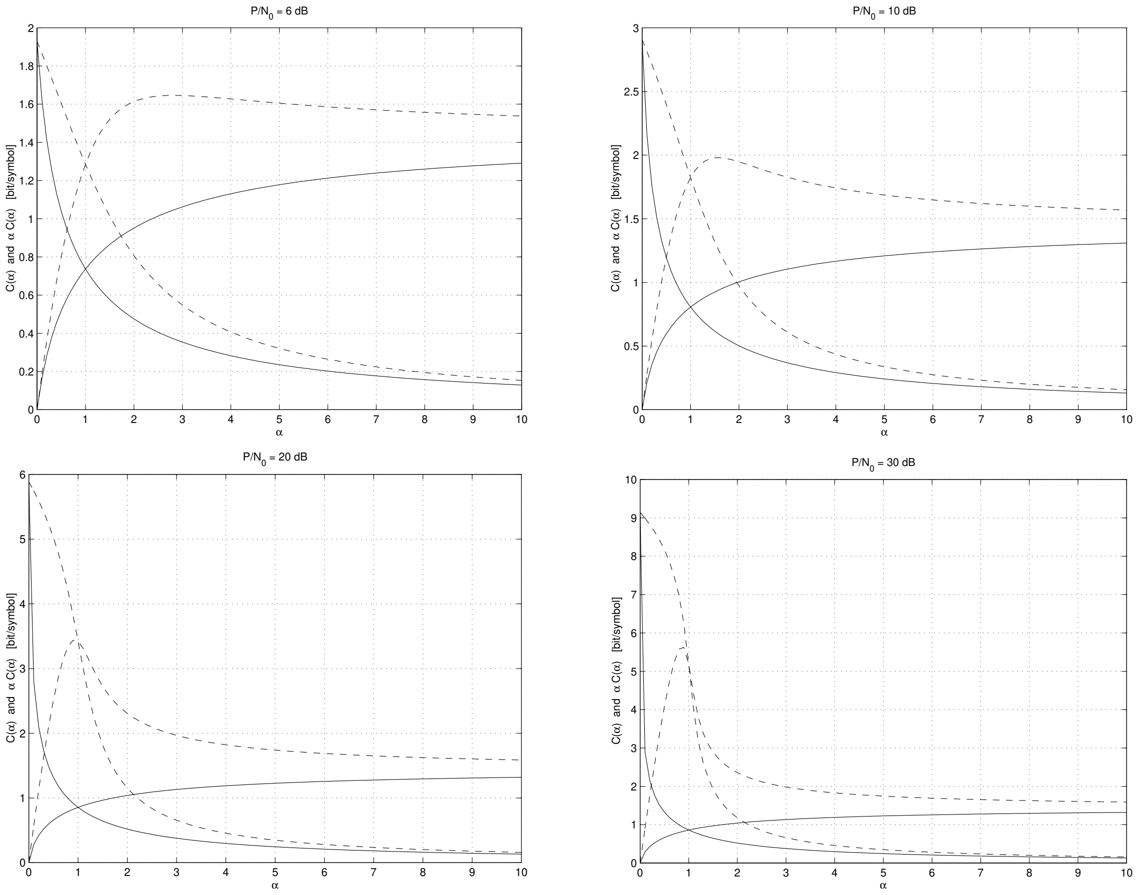

Figure 3: Plot of $C(\alpha)$ and $\alpha\, C(\alpha)$ versus α for the MMSE (dashed) and SUMF (solid) receiver for $\eta = 6, 10, 20, 30$ dB.

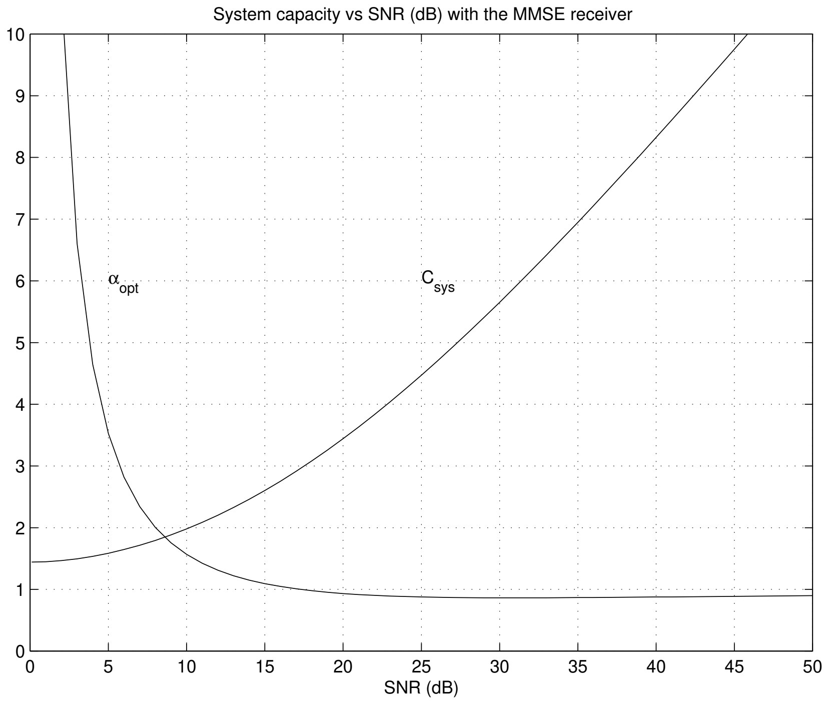

Figure 4: System capacity $C_{\rm sys}$ and optimal α versus η for the MMSE receiver.

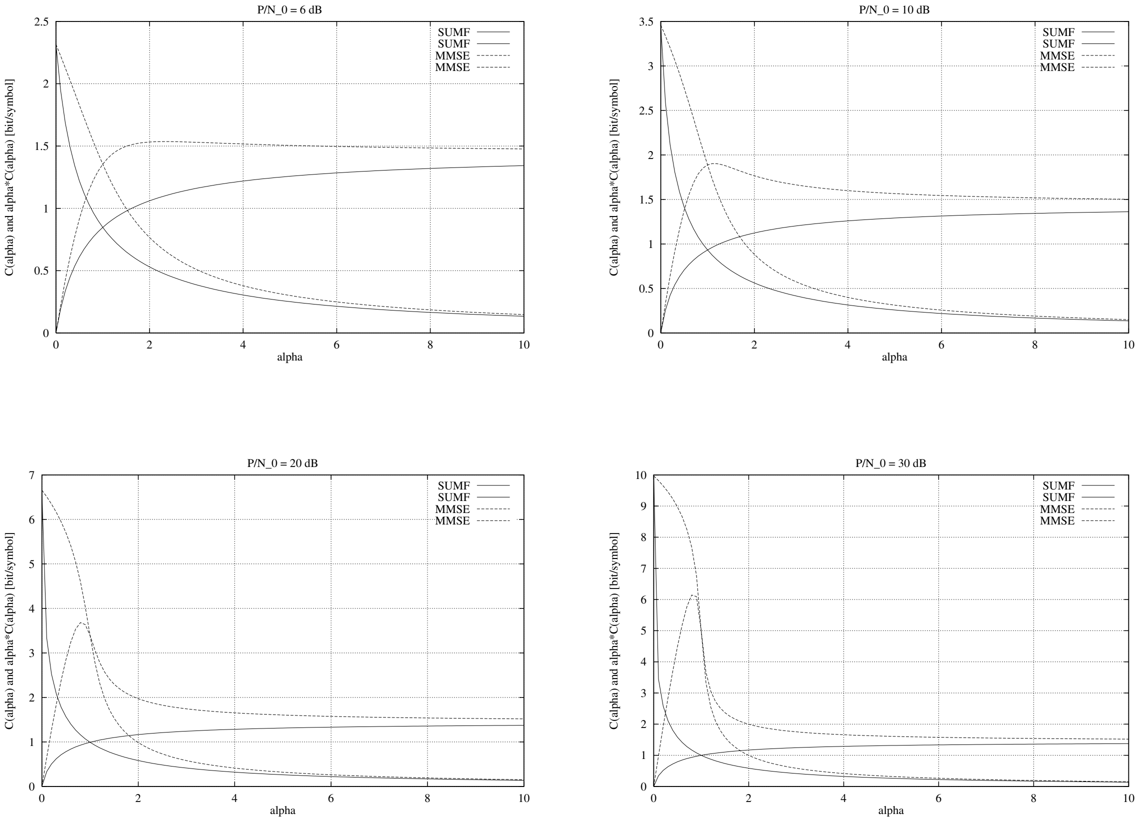

Figure 5: Plots of $C(\alpha)$ and $\alpha\, C(\alpha)$ versus α for the SUMF (solid) and MMSE (dashed) receiver for $\eta = 6, 10, 20, 30$ dB.

References

[1] E. Biglieri, J. Proakis, and S. Shamai (Shitz), "Fading channels: Information-theoretic and communications aspects," *IEEE Trans. Inform. Theory*, Vol. 44, No. 6, pp. 2619–2692, October 1998.

[2] G. Caire, G. Taricco, and E. Biglieri, "CDMA system design through asymptotic analysis," submitted for publication, 1999.

[3] D. Tse and S. Hanly, "Linear multiuser receivers: Effective interference, effective bandwidth and capacity," *IEEE Trans. on Inform. Theory*, Vol. 45, No. 2, pp. 641–657, March 1999.

[4] D. Tse and S. V. Hanly, "Network capacity, power control, and effective bandwidth," in: H. Vincent Poor and Gregory W. Wornell (Eds.), *Wireless Communications : Signal Processing Perspectives*. Prentice Hall Signal Processing Series, 1998.

[5] S. Verdú and S. Shamai (Shitz), "Spectral efficiency of CDMA with random spreading," *IEEE Trans. on Inform. Theory*, Vol. 45, No. 2, pp. 622–640, March 1999.

[6] S. Verdú, *Multiuser Detection*. New York: Cambridge University Press, 1998.