Number fields and Modulations for the Fading Channel

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Abstract

In this talk we present an interesting application of algebraic number theory to the design of signal constellations for the fading channel. The pourpose is to outline some open mathematical problems, the solution of which could improve our theoretical understanding and give us new tools for practical applications.

The channel model is the following. Let $\mathbf{x} = (x_1, x_2, \ldots x_n)$ denote a transmitted signal vector from a given n-dimensional constellation S. Received signal samples are then given by $y_i = g_i x_i + n_i$ for $i = 1, 2, \ldots n$. Here, g_i are (real) independent Rayleigh-distributed random variables with unit second moment (i.e., $E[g_i^2] = 1$) representing the fading coefficients and n_i are real Gaussian random variables with mean zero and variance $N_0/2$ representing the additive noise. We also denote $\mathbf{g} = (g_1, g_2, \ldots g_n)$, $\mathbf{n} = (n_1, n_2, \ldots n_n)$, and $\mathbf{y} = (y_1, y_2, \ldots y_n)$ and write $\mathbf{y} = \mathbf{g} \odot \mathbf{x} + \mathbf{n}$. Note that when all the $g_i = 1$ we fall back to the well known Additive White Gaussian Noise (AWGN) channel.

For maximum-likelihood (ML) detection we compute the sample metrics

$$m(\mathbf{x}|\mathbf{y}, \mathbf{g}) = ||\mathbf{y} - \mathbf{g} \odot \mathbf{x}||^2 \quad \forall \quad \mathbf{x} \in S \tag{1}$$

where $|| \cdot ||^2$ is the standard Euclidean norm, and decide for the signal $\widehat{\mathbf{x}}$ attaining the minimum value of $m(\mathbf{x}|\mathbf{y}, \mathbf{g})$.

It can be shown that the error probability of such a system is essentially dominated by the following parameters: i) the *modulation diversity* L: the minimum number of *different* components between any two distinct points in the signal set; ii) the *minimum L-product distance* $d_{p,\min}^{(L)} = \min_{\mathbf{x}, \widehat{\mathbf{x}}} \prod_{x_i \neq \widehat{x}_i} |x_i - \widehat{x}_i|$ of the constellation. Good signal sets maximize L and $d_{p,\min}^{(L)}$ under an average energy constraint $E[||\mathbf{x}||]$ =constant. For $L \to \infty$ the error rate performance of a signal constellation over a fading channel approaches the one over AWGN channel.

In [3] it is shown that by the *canonical embedding* σ of an algebraic number field K of degree n it is possible to construct an n-dimensional lattice $\Lambda_{n,L} = \sigma(K)$ with diversity $L = r_1 + r_2$, where (r_1, r_2) is the signature of the field. For totally real fields we have full-diversity lattices, $L = n$. For totally complex fields we have lattices with diversity $L = n/2$. Only in the totally real case also $d_{p,\min}^{(L)} = 1$ corresponding to the image in the Euclidean space of the field units. In all other cases no relation was found for $d_{p,\min}^{(L)}$.

Using this approach, lattices from minimal absolute discriminant fields were first constructed. These fields are known in the mathematical literature up to dimension 8 [2]. The resulting lattices were discarded for practical use since no fast binary labeling algorithm was found. Another negative feature for application of these signal constellations on AWGN channels is their low packing density.

Applying Craig's technique [1] the root lattices $D_{4,2}$, $E_{6,3}$, $E_{8,4}$, $K_{12,6}$, $\Lambda_{16,8}$ and $\Lambda_{24,12}$ were constructed in [3] by the canonical embedding of particular ideals of some cyclotomic fields. These lattices overcome the two problems of the previous family of lattices. First, they are sub-lattices of a rotated version of the cubic lattice Z^n and admit a code construction so that the binary labeling is simple. Second, when used over AWGN channel, they have very good performance due to their high packing density.

From the previous family of lattices, the next step was to consider rotated cubic lattice constellations $Z_{n,L}$. Different approaches are analyzed [4].

- Construction of $Z_{n,n/2}$ lattices from the canonical embedding of cyclotomic fields of degree $n = 2^{e_1}3^{e_2}$, $e_1, e_2 = 0, 1, 2, \ldots$ and $n \geq 4$.
- Construction of some $Z_{n,n}$ lattices from the canonical embedding of ideals of the fields $\mathbf{Q}(2\cos(2\pi/N))$ with $n = \phi(N)/2$ (ϕ is the Euler function).
- Direct optimization of $d_{p,\min}^{(n)}$ of particular families of rotation matrices giving $Z_{n,n}$ lattices.

An open problem is a complete classification of these rotated cubic lattices. A nice property of these constellations is their extremly simple binary labeling. A Gray labeling (1 bit changes in the label moving to each nearest neighbour) can be used and we verified experimentally that this labeling improves the bit error probability compared to other types of mappings.

We conclude with some issues on the decoding algorithm. Minimizing (1) for arbitrary constellations can only be accomplished by exhaustive search through all the points. In the case of lattice constellations a simple modification of the minimum Euclidean distance lattice decoding algorithm enables to speed up the decoder [5]. With the present algorithm we are able to decode lattice codes up to dimension 32, due to the large computational complexity. Improvements of this algorithm for some particular families of lattices should be investigated.

References

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