Optimal Energy Transfer in Bandlimited Communication Channels

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Résumé

On considere le problème de calculer le maximum du rapport d'énergie transférée par des signaux de durée limitée transmis sur des canaux linéaires. Le cas des canaux lineaires avec fonction de transfert rationelle est introduit et une procedure de solution analytique du problème de optimization est décrit. Le rapport d'énergie transférée dans un intervalle de temps limité est calculé dans des cas d'intérêt. Deux mesures de performance — rapport d'énergie transférée et rapport d'énergie IES — sont calculé pour les signaux optimaux et comparé avec les signaux rectangulaires et sinusoidelles utilisée couramment pour évaluer les gains que l'on peut atteindre. Le problème de maximization est resolu pour un interval de sortie $J = I = [0, T]$ et sa version retardée $J = [t_0, t_0 + T] = t_0 + I$.

Abstract

Maximization of the energy transfer ratio for time-limited signals over linear channels is considered. The case of linear channels with rational transfer function is addressed and a general procedure for the analytic solution of the optimization problem is outlined. The maximum energy transfer ratio over a fixed time interval is evaluated in some cases of interest. Two performance metrics — the energy transfer ratio and the energy ISI ratio — are evaluated for the optimal signals and compared to those of commonly used rectangular and sinusoidal pulses in order to determine the achievable gain. The maximization problem is solved for an output interval $J = I = [0, T]$ and for its delayed version $J = [t_0, t_0 + T] = t_0 + I$.

1 Introduction

According to standard Fourier analysis, time-limited signals passing through band-limited channels are spread in time. The energy dispersion introduced is a source of performance degradations which are partially recovered in many applications by equalization techniques. These counter-measures would certainly be more efficient if used jointly with signals which have a minimal energy stretched along the signaling interval by filtering, or which have their maximum energy transferred over the signaling interval. This optimization problem attracted the attention of many researchers early in the fifties because of its implications for digital communications, system theory and related fields. Much of this work was devoted to the ideal low-pass filter on account of its natural connection with the representation of band-limited signals [11, 6, 7]. In these applications, the energy spread outside the interval $I = [0, T]$ produces disturbances known as intersymbol interference (ISI). Hence, the maximum signal energy transferred at the filter output is an important issue in the design of good signal sets for digital transmissions. This energy transfer is measured as the output-to-input signal energy ratio over potentially different time intervals (in this paper we denote by I and J the input and output time interval, respectively). Most works in this area refer to an unlimited output time interval $J = (-\infty, \infty)$ and, according to this assumption, Slepian [11] solved the problem for the ideal low-pass channel. Specifically, he showed that the optimal pulses are related to prolate spheroidal wave functions, and that these signals are also optimal over the limited output time interval $J = I$. This property is not true for all channels, as we will show in this paper. Subclasses of rational transfer function channels have been considered as well as the ideal low-pass channel. For instance, in [5] the optimal waveforms are found for the single-pole low-pass filter and in [2, 3] the optimal waveforms are found for Butterworth filters referring to the output time interval $(-\infty, \infty)$. In these papers special analytic techniques were developed which can be derived from Youla's general approach [12].

Undoubtedly, the case of limited output time interval J is the most interesting because gives the minimum achievable ISI for a fixed channel bandwidth W and symbol time duration T. It has, however, received little attention probably due to the fact that analytical solutions seemed unlikely. In this paper, the maximization problem is addressed for an output interval $J = I = [0, T]$ and for its delayed version $J = [t_0, t_0 + T] = t_0 + I$. The introduction of a delay t_0 may lead to significant reduction of the ISI without increasing the complexity of the system. Delays

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often occur in digital communications and are managed routinely. The computation of the minimum time offset t_0 yielding the maximum transfer of energy will be carried out as part of the optimization process.

2 General results

In this work we consider square-integrable functions $y(t) \in L^2(R)$ over the real axis $R = (-\infty, \infty)$. The energy $\mathcal{E}_J[y]$ of $y(t)$ included in the interval J is given by the integral $\mathcal{E}_J[y] = \int_J y(t)^2 \, dt$. Whenever $y(t)$ is the output of a linear channel with impulse response $h(t, \tau)$ and input signal $x(t) \in L^2(I)$, we have $y(t) = \int_I h(t,\tau)x(\tau) \, d\tau$, $t \in R$. The energy of $y(t)$ included in $J \subset R$ can be expressed as a quadratic integral form in $x(t)$

$$\mathcal{E}_J[y] = \int_J y(t)^2 \, dt = \int_I \int_I \mathcal{K}_J(t,\tau)x(t)x(\tau)dtd\tau \qquad (1)$$

where the energy kernel is defined as the integral

$$\mathcal{K}_J(t,\tau) = \int_J h(\theta,t)h(\theta,\tau) \, d\theta \qquad (2)$$

The filtered signals are spread in time, and the resulting energy dispersion is usually an unwanted side-effect.

Definition 1 *The* energy transfer ratio *and the* energy ISI ratio *from I to J for a channel with impulse response $h(t,\tau)$, input $x(t)$, $t \in I$, and output $y(t)$, $t \in J$, are defined respectively as*

$$\lambda[x, J, h] \; = \; \frac{\mathcal{E}_J[y]}{\mathcal{E}_I[x]} \quad \textit{and} \quad \eta[x, J, h] \; = \; \frac{\mathcal{E}_R[y] - \mathcal{E}_J[y]}{\mathcal{E}_J[y]}$$

When $J = I$ the energy transfer ratio is defined in such a way that the greater the ratio, the greater the energy passed through the filter. Hence, for channels with additive noise, optimization of this figure increases the signal-to-noise ratio. The energy ISI ratio, on the other hand, is a measure of the amount of ISI because it directly compares the energy included in the signaling time interval with the energy spread outside it.

Problem 1 *Given a linear channel with impulse response $h(t,\tau)$, find a signal $x(t) \in L^2(I)$ with fixed energy $\mathcal{E}_I[x] = \mathcal{E}$ which gives the maximum $\lambda_{\rm opt}$ of $\lambda[x, J, h]$.*

Necessary, and in many cases sufficient, conditions characterizing any solution $x(t)$ are already known [5, 8].

Proposition 2.1 *Every solution of* Problem 1 *is a signal $x(t)$ satisfying the following Fredholm integral equation of the second kind*

$$\int_I \mathcal{K}_J(t,u)x(u)du \; = \; \lambda x(t) \qquad t \in I \qquad (3)$$

with a positive definite kernel, as defined in equation (2). The maximum eigenvalue λ gives the maximum energy transfer ratio.

When $J = t_0 + I$, the optimization procedure involves two steps. First, t_0 is fixed and the optimum given by Proposition 2.1 is computed, then the best t_0 is found using the following proposition.

Proposition 2.2 *The solution of* Problem 1 *when $J = t_0 + I$ is given by a normalized eigenfunction $x(t)$ of $\mathcal{K}_J(t,u)$ defined in (2):*

$$\int_I \mathcal{K}_J(t,\tau)x(\tau)d\tau \; = \; \lambda x(t) \qquad t \in I \qquad (4)$$

and t_0 is the minimum nonnegative value obtained from the condition

$$y(t_0)^2 \; = \; y(t_0 + T)^2 \qquad (5)$$

It can easily be seen that for time-invariant linear channels with time-symmetric (even) impulse response the maximum energy transfer over a finite output interval $J = t_0 + I$ occurs for $t_0 = 0$. Although these transfer functions are not physically realizable, they include important filters such as the ideal low-pass filter. Therefore Slepian's optimal solution for $J = I$ is also optimal for $J = t_0 + I$ with $t_0 = 0$.

3 Rational transfer functions

We now consider channels with a rational transfer function and solve Problem 1 for a finite output time interval $J = [0, T_o]$. Let the filter rational transfer function be given as a Laplace transform:

$$H(s) = \mathcal{L}\{h(\tau)\} = \frac{1 + h_1 s + h_2 s^2 + \ldots + h_m s^m}{1 + g_1 s + g_2 s^2 + \ldots + g_n s^n} = \frac{M(s)}{P(s)}$$

with $n > m$, and let us assume that $H(s)$ has simple poles $\alpha_1, \ldots, \alpha_n$. The impulse response is consequently

$$h(t) = u(t) \sum_{i=1}^{n} A_i e^{\alpha_i t}$$

where $u(t)$ denotes the unit step function.
Our aim is to find the sequences of eigenfunctions $x_n(t)$ and eigenvalues λ_n which satisfy the Fredholm integral equation of Problem 1. The case of an infinite output time interval has been settled by several authors [10, 12, 2, 5] who developed various techniques based on the common principle that commuting operators admit the same set of eigenfunctions [1]. In particular, if $H(s)$ has simple poles, Youla's solution [12] exploits the rational property of the transfer function and halves the labor of solving a transcendental equation which cannot be further simplified. Although Youla's simplification does not apply to finite J, most of the arguments still work if combined with Franks' approach [3]. The method we propose can be summarized in a three step program.

Step 1: Find a differential operator commuting with the Fredholm integral operator.

Step 2: Let $K(s) = H(s)H(-s) = N(s^2)/D(s^2)$. Obtain a candidate eigenfunction solving a differential equation

$$\left(D(\frac{d^2}{dt^2}) - \frac{1}{\lambda} N(\frac{d^2}{dt^2}) \right) x(t) = 0$$

Let $\pm\sigma_1, \ldots, \pm\sigma_n$ be the $2n$ simple roots of the characteristic equation

$$D(s^2) - \frac{1}{\lambda}N(s^2) = 0$$

then the eigenfunctions for $\mathcal{K}_J(t,\tau)$ are of the form

$$x(\tau) = \sum_{i=1}^{n}[C_\ell e^{\sigma_i t} + D_\ell e^{-\sigma_i t}] \qquad t \in I \qquad (6)$$

Step 3: Obtain a transcendental equation for the eigenvalues substituting the candidate eigenfunction into (3). Substitute the candidate eigenfunction of the form (6) into (3) and obtain an homogeneous linear system of $2n$ equations in the $2n$ unknowns $C_\ell, D_\ell,\ \ell = 1, \ldots, n$. Looking for non-zero solutions, the determinant of the coefficient matrix must be zero, a condition which yields the transcendental equations for the eigenvalues. Finally, the non-zero solution of the system gives the coefficients characterizing the eigenfunctions up to a scale factor. All the mathematical details of this procedure can be found in [4].

Example – As an illustrative example, we apply the above procedure to the single-pole linear channel with transfer function

$$H(s) = \frac{1}{1 + s/(2\pi W)}$$

where W is the 3-dB bandwidth. The impulse response for this channel is

$$h(t) = 2\pi W u(t) e^{-2\pi W t}$$

where $u(t)$ denotes the unit step function. The Laplace transform of the energy kernel for $J = R$ is the ratio of the two polynomials:

$$N(s^2) = 1 \qquad \text{and} \qquad D(s^2) = 1 - \frac{s^2}{(2\pi W)^2}$$

Let us assume that input signals have duration T and the finite output time interval is $J = [0, T_o]$. Hence the energy kernel

$$\mathcal{K}(t,\tau) = \pi W \left[e^{-2\pi W|t-\tau|} - e^{2\pi W(t+\tau-2T_o)}\right]$$

is certainly not time-invariant. However, we may specialize general formulas or apply Franks' method [5]. In both cases, taking the second derivative of the equation

$$\lambda\psi(t) = \int_0^T \psi(\tau)\pi W \left[e^{-2\pi W|t-\tau|} - e^{2\pi W(t+\tau-2T_o)}\right] d\tau \qquad (7)$$

we obtain the differential equation

$$\lambda\frac{d^2\psi(t)}{dt^2} = -(2\pi W)^2\psi(t) + \lambda(2\pi W)^2\psi(t)$$

The characteristic equation of this linear differential equation is

$$1 - \frac{s^2}{(2\pi W)^2} - \frac{1}{\lambda} = 0$$

The roots are purely imaginary $\sigma = \pm 2\pi j W z$ with $z = \sqrt{1/\lambda - 1}$ and the real form of the eigenfunctions is

$$\psi(t) = C\cos(2\pi W z t) + D\sin(2\pi W z t)$$

Substituting one of these expressions into (7), we obtain for z the following transcendental equation:

$$\left[e^{-4\pi W(T_o - T)}(1 + z_n^2) + (1 - z_n^2)\right] \cdot$$
$$\cdot \sin(2\pi W T z_n) + 2z_n\cos(2\pi W T z_n) = 0$$

Note that when $T_o \to \infty$, we obtain Youla's solution [12] $z_{2n-1} = \cot(\pi W T z_{2n-1})$, $z_{2n} = -\tan(\pi W T z_{2n})$ for $n = 1, 2, \ldots$. When $T_o = T$, we obtain $\tan(2\pi W T z_n) = -z_n$. The real eigenfunctions are

$$\psi_n(t) = A[z_n\cos(2\pi W T z_n) + \sin(2\pi W T z_n)]$$

where the normalizing factor A can be easily computed. Once z_n is known the corresponding eigenvalue λ_n is obtained as $\lambda_n = 1/(1 + z_n^2)$.

Let us now consider the case with the time offset $J = [t_0, t_0 + T]$. Consequently, $\mathcal{K}_J(t,\tau)$ is not time-invariant but it can be shown that the eigenfunctions have the same form as for $t_0 = 0$ and the eigenvalues are obtained from the same equation with t_0 as a parameter. The received signal $y(t)$ evaluated at times t_0 and $t_0 + T$ gives

$$\begin{cases} y(t_0) &= A\sin(2\pi z_n W t_0) \\ y(t_0 + T) &= A e^{-2\pi W t_0}\sin(2\pi z_n W T) \end{cases}$$

Hence we obtain the equations

$$[(1 - z_n^2) + e^{-4\pi W t_{0n}}(1 + z_n^2)]\tan(2\pi z_n W T) + 2z_n = 0$$

$$e^{-4\pi W t_{0n}}\sin^2(2\pi z_n W T) - \sin^2(2\pi z_n W t_{0n}) = 0$$

where we wrote t_{0n} for t_0 as the optimal delay which depends on the order n of the eigenvalue $\lambda_n = 1/(1 + z_n^2)$. These two equations suggest an iterative method for calculating t_{0n} and z_n. Starting with a tentative value of $z_n^{(0)}$, near to the value obtained for $t_{0n} = 0$, from the first equation we estimate $t_{0n}^{(1)}$ as

$$t_{0n}^{(1)} = \frac{-1}{4\pi W}\ln\left[\frac{-2z_n^{(0)}\cot(2\pi z_n W T) - (1 - [z_n^{(0)}]^2)}{1 + (z_n^{(0)})^2}\right]$$

Then we substitute into the second equation and we solve for $z_n^{(1)}$. We compute $t_{0n}^{(2)}$ using the first equation and so on.

Results for $2WT = 1, 2, 4$ are reported in Table 1, where we considered rectangular and sinusoidal pulses as well as optimal signals. Both the energy transfer ratio and the energy ISI ratio have been evaluated for different combinations of the parameters. The results show the performance enhancement of optimal vs. standard waveforms. J is chosen as $t_0 + I$ in order to maximize $\mathcal{E}_J$ with respect to t_0. The results show that the performance of the sinusoidal signal set is very close to that of the optimal set for $c = 2$ and 4 in terms of $\mathcal{E}_R$ and $\mathcal{E}_J$. A larger difference is observed for the rectangular signal set. Focusing on the

Waveforms	i	λ_i	t_{0i}	$\mathcal{E}_R$	$\mathcal{E}_J$	η (%)
$2WT = 1$						
Rect. pulses	1			0.695	0.549	26.5
Sin(πt)	1			0.666	0.579	14.9
Opt. on I	1	0.6172		0.680	0.6172	10.2
Opt. on $t_0 + I$	1	0.6683	0.121	0.7027	0.6586	6.7
Opt. on R	1	0.7105		0.7105	0.5824	22.0
$2WT = 2$						
Rect. pulses	1			0.8410	0.7620	10.4
	2			0.5497	0.4830	13.8
Sin(πt)	1			0.8510	0.8254	3.1
Sin(2πt)	2			0.5794	0.5398	7.3
Opt. on I	1	0.8410		0.8598	0.8410	2.2
	2	0.5609		0.5969	0.5609	6.4
Opt. on $t_0 + I$	1	0.8591	0.133	0.8650	0.8516	1.6
	2	0.5860	0.117	0.6139	0.5860	4.8
Opt. on R	1	0.8718		0.8718	0.8233	5.9
	2	0.6172		0.6172	0.5395	14.4
$2WT = 4$						
Rect. pulses	1			0.9204	0.8806	4.5
	2			0.7619	0.7224	5.5
	3			0.6164	0.5832	5.7
	4			0.4831	0.4496	7.4
Sin(πt)	1			0.9500	0.9456	0.46
Sin(2πt)	2			0.8255	0.8127	1.57
Sin(3πt)	3			0.6767	0.6583	2.8
Sin(4πt)	4			0.5398	0.5199	3.8
Opt. on I	1	0.9490		0.9526	0.9490	0.4
	2	0.8220		0.8329	0.8220	1.3
	3	0.6703		0.6870	0.6703	2.5
	4	0.5309		0.5499	0.5309	3.6
Opt. on $t_0 + I$	1	0.9519	0.067	0.9553	0.9519	0.36
	2	0.8312	0.067	0.8405	0.8312	1.1
	3	0.6824	0.067	0.6971	0.6824	2.2
	4	0.5417	0.067	0.5600	0.5417	3.4
Opt. on R	1	0.9553		0.9553	0.9441	1.2
	2	0.8410		0.8410	0.8094	3.9
	3	0.6982		0.6982	0.6560	6.4
	4	0.5609		0.5909	0.5205	7.7

Table 1: Energy transfer for a single pole low-pass filter with "optimal", "rectangular" and 'sinusoidal" signals.

energy ISI ratio, we note a larger advantage in using the optimal signal set.

Numerical computation of eigenvalues and eigenfunctions is generally a critical issue because of possible numerical instability. This motivates resorting to analytic solutions in special cases to check the accuracy of the numerical results obtained. The computation of the eigenvalues is amenable to the solution of a transcendental equation.

4 Conclusions

In this paper we considered the problem of computing sets of signals which convey the maximum amount of energy included in the signaling time interval. Reference has been made to the class of linear filters with rational transfer functions. We have shown that the computation of the eigenvalues can be traced back to the solution of a transcendental equation and the eigenfunctions are linear combinations of possibly complex exponential time functions with exponent coefficients which are algebraic functions of the eigenvalues. We have also computed eigenvalues and eigenfunctions by direct numerical integration and numerical optimization. The agreement between the results obtained with the two methods validates the "numerical" technique, which is preferable or unavoidable when dealing with complex rational transfer functions.

Knowing the form of the eigenfunctions is also important for practical implementation. It may be easier to generate linear combinations of exponential and trigonometric functions rather than to interpolate inaccurate samples obtained by numerical integrations.

Finally, we introduced the energy ISI ratio, a second figure of merit besides the classical energy transfer ratio, which is significant if the optimization aims at reducing ISI.

References

[1] D. Courant and D. Hilbert, *Mathematical Methods in Physics.* Englewood Cliffs: Prentice-Hall, 1955.

[2] M. Elia and M. T. Galizia Angeli, "Autofunzioni di operatori lineari integrali associati ai filtri di Butterworth," VII GMEL, Coimbra, September 1985.

[3] M. Elia and M. T. Galizia Angeli, "Eigenfunction of integral linear operators associated to second order filters," Proceedings of IASTED Int. Symp. MIC '87, Grindelwald, Switzerland, pp. 396-399.

[4] M. Elia, G. Taricco, E. Viterbo, "Optimal Energy Transfer in Bandlimited Communication Channels," submitted to *IEEE Trans. On Inform. Theory*, October 1996.

[5] L. E. Franks, *Signal Theory*, New Jersey: Prentice Hall, 1969.

[6] H.J. Landau and H.O. Pollack, "Prolate spheroidal wave functions, Fourier analysis and uncertainty II," *Bell Syst. Tech. J.*, 1961, Vol. 40, pp. 65-84.

[7] H.J. Landau and H.O. Pollack, "Prolate spheroidal wave functions, Fourier analysis and uncertainty III," *Bell Syst. Tech. J.*, 1962, Vol. 41, pp. 1295-1336.

[8] P.G. Petrovsky, *Lectures on the Theory of Integral Equations.* Moscow: MIR, 1971.

[9] B. van der Pol and H. Bremmer, *Operational Calculus Based on the Two Sided Laplace Transform.* Cambridge Univ. Press, Cambridge, Eng., 1955.

[10] D. Slepian, "Some comments on Fourier analysis, uncertainty and modeling," *SIAM Review*, July 1982, Vol. 25, No. 3, pp. 379-393.

[11] D. Slepian and H.O. Pollack, "Prolate spheroidal wave functions, Fourier analysis and uncertainty I," *Bell Syst. Tech. J.*, 1961, Vol. 40, pp. 43-64.

[12] D.C. Youla, "The solution of a homogeneous Wiener-Hopf integral equation occurring in the expansion of second-order stationary random functions," *IRE Trans. on Information Theory*, Sept. 1957, pp.187-193.