The Golden Code: A 2×2 Full-Rate Space-Time Code with Non-Vanishing Determinants

Jean-Claude Belfiore, Ghaya Rekaya
ENST
46 rue Barrault, 75013 Paris, France
{belfiore,rekaya}@enst.fr

Emanuele Viterbo[1]
Politecnico di Torino
C. Duca degli Abruzzi 24, Torino, Italy
e-mail: emanuele.viterbo@polito.it

Abstract — **In this paper we present the Golden code for a 2×2 MIMO system. This is a full-rate 2×2 linear dispersion algebraic space-time code with unprecedented performance based on the Golden number $\frac{1+\sqrt{5}}{2}$.**

Full rate and full diversity algebraic codes for 2×2 coherent MIMO systems, were first constructed in [2], using number theoretical methods. This approach was later generalized for any number of transmit antennas M [5, 4]. The above constructions satisfy the *rank criterion* and attempt to maximize, for a fixed signal set S, the *coding advantage*. A general family of 2×2 full-rank and full-rate linear dispersion space-time block codes (LD-STBC), based on quaternion algebras, was given in [1, 6].

Let $\mathbb{K} = \mathbb{Q}(\theta)$ be a quadratic extension of $\mathbb{Q}(i)$, we define the *infinite code* $\mathcal{C}_\infty$ as the set of matrices of the form

$$\mathcal{C}_\infty = \left\{ \mathbf{X} = \left[\begin{array}{cc} a+b\theta & c+d\theta \\ \gamma(c+d\bar{\theta}) & a+b\bar{\theta} \end{array} \right] : a,b,c,d \in \mathbb{Z}[i] \right\}$$

$\mathcal{C}_\infty$ is clearly a linear code, i.e., $\mathbf{X_1}+\mathbf{X_2} \in \mathcal{C}_\infty$ for all $\mathbf{X_1}, \mathbf{X_2} \in \mathcal{C}_\infty$. The *finite code* $\mathcal{C}$ is obtained by limiting the *information symbols* to $a,b,c,d \in S \subset \mathbb{Z}[i]$, where we assume the signal constellation S to be a 2^b–QAM, with in-phase and quadrature components equal to $\pm 1, \pm 3, \ldots$ and b bits per symbol.

The code $\mathcal{C}_\infty$ is a discrete subset of a *cyclic division algebra over* $\mathbb{Q}(i)$, obtained by selecting $\gamma \neq N_{\mathbb{K}/\mathbb{Q}(i)}(x)$ for any $x \in \mathbb{K}$ [1, 6]. A division algebra naturally yields a structured set of invertible matrices that can be used to construct square LD-STBC since for any codeword $\mathbf{X} \in \mathcal{C}_\infty$ the rank criterion is satisfied as $\det(\mathbf{X}) \neq 0$.

We define the *minimum determinant* of $\mathcal{C}_\infty$ as

$$\delta_{\min}(\mathcal{C}_\infty) = \min_{\mathbf{X} \in \mathcal{C}_\infty, \mathbf{X} \neq 0} |\det(\mathbf{X})|^2 \qquad (1)$$

and the minimum determinant of the finite code $\mathcal{C}$ as

$$\begin{aligned} \delta_{\min}(\mathcal{C}) &\triangleq \min_{\mathbf{X_1}, \mathbf{X_2} \in \mathcal{C}, \mathbf{X_1} \neq \mathbf{X_2}} |\det(\mathbf{X_1} - \mathbf{X_2})|^2 \\ &= 4\delta_{\min}(\mathcal{C}_\infty) \qquad (2) \end{aligned}$$

Minimum determinants of $\mathcal{C}_\infty$ in all previous constructions [2, 5, 4, 6] are non-zero, but vanish as the spectral efficiency b of the signal constellation S is increased. This problem appears because either transcendental elements or algebraic elements with a too high degree are used to construct the division algebras. Non-vanishing determinants may be of interest, whenever we want to apply some outer block coded modulation scheme, which usually entails a signal set expansion, if the spectral efficiency has to be preserved.

In order to obtain *energy efficient* codes we need to construct a lattice $M\mathbb{Z}[i]^2$, a rotated version of the complex lattice $\mathbb{Z}[i]^2$, where M is a complex unitary matrix, so that there is no shaping loss in the signal constellation emitted by the transmit antennas. This additional property was never considered before and is the key to the improved performance our codes.

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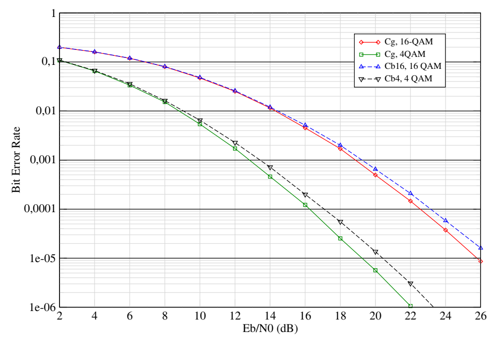

Figure 1: Performance comparison of the new codes vs. those of [2] and [3]

The algebraic construction yields codewords of the Golden code of the form

$$\frac{1}{\sqrt{5}} \left[\begin{array}{cc} \alpha(a+b\theta) & \alpha(c+d\theta) \\ i\bar{\alpha}(c+d\bar{\theta}) & \bar{\alpha}(a+b\bar{\theta}) \end{array} \right] \qquad a,b,c,d \in \mathbb{Z}[i]$$

where $\alpha = 1+i(1-\theta)$, $\theta = \frac{1+\sqrt{5}}{2}$ and $\bar{\theta} = \frac{1-\sqrt{5}}{2}$. We show that the Golden code has non-vanishing $\delta_{\min}(\mathcal{C}_\infty) = 1/5$, hence $\delta_{\min}(\mathcal{C}) = 1/5$ for any size of the signal constellation. Fig. 1 shows how the Golden code outperforms all previous constructions.

References

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