Robust Codes for 2×2 MIMO Block Fading Channels

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***Abstract*— Golden space-time trellis coded modulation (GST-TCM) scheme was proposed in [1] for a high rate 2×2 multiple-input multiple-output (MIMO) system over slow fading channels. In this paper, we present the design criteria of GST-TCM over general block fading channels, where the channel matrix is constant over a fraction of the codeword length and varies from one fraction to another independently. However, the code construction and optimization can be difficult to implement. We therefore analyze the performance of GST-TCM for slow fading over block fading channels. The impact of the block fading channel on the code performance is analyzed using a truncated Union Bound technique. We finally show both analytically and by simulation that the GST-TCM designed for slow fading channels are indeed robust to various channel conditions. This feature is particularly useful for transmission over multipath channels using multicarrier modulation such as OFDM.**

I. Introduction

The Golden code was proposed in [2] as a full rate and full diversity code for 2×2 multiple-input multiple-output (MIMO) systems with *non-vanishing minimum determinant* (NVD). It was shown in [3] how this property guarantees to achieve the diversity-multiplexing gain trade-off.

In order to enhance the coding gain, a first attempt to concatenate the Golden code with an outer trellis code was made in [4]. However, the resulting *ad hoc* scheme suffered from a high trellis complexity.

In [1, 6], a Golden space-time trellis coded modulation (GST-TCM) scheme was designed for slow fading channels. The NVD property of the inner Golden code is essential for a TCM scheme where a constellation expansion is usually required. This property guarantees that it will not suffer from a reduction of the minimum determinant [2].

The systematic design approach proposed in [1, 6], is based on set partitioning of the Golden code in order to increase the minimum determinant. An outer trellis code is used to increase the Hamming distance between the codewords. The Viterbi algorithm is used for trellis decoding, where the branch metrics are computed by using a lattice sphere decoder [8] for the inner Golden code.

In this paper, we extend the analysis of the GST-TCM scheme to *block fading* channels [5]. The block fading channel model represents a simple and powerful tool to describe a variety of fading channels ranging from fast to slow. We show the design criteria for the block fading channel and note that code construction and optimization can be hard to realize. Hence, we derive the performance analysis for GST-TCM over block fading channels. The impact of the block fading channel on the code performance is analyzed using a truncated union bound technique. We finally show by simulation that the GST-TCM designed for slow fading channels are indeed robust to various channel conditions.

The rest of the paper is organized as follows. Section II introduces and motivates the system model for block fading channels. Section III presents the design criteria of GST-TCM over block fading channels. In Section IV, the impact of various channel conditions on the performance of GST-TCM designed for slow fading is analyzed. Section V shows simulation results. Conclusions are drawn in Section VI.

The following notations are used in the paper. Let T denote transpose and $\dagger$ Hermitian transpose. Let $\mathbb{Z}$, $\mathbb{C}$ and $\mathbb{Z}[i]$ denote the ring of rational integers, the field of complex numbers, and the ring of Gaussian integers, respectively, where $i^2 = -1$. Also, we let $\min(a, b)$ represent the minimum of a and b.

II. System Model and Block Fading

Let us consider a 2×2 MIMO system ($n_T = 2$ transmit and $n_R = 2$ receive antennas) over a slow fading channel where the Golden code $\mathcal{G}$ is used. The 2×2 Golden codeword $X \in \mathcal{G}$ is transmitted over two channel uses where the channel matrix H is constant and we receive

$$Y = HX + Z \tag{1}$$

where Z is the complex white Gaussian noise matrix 2×2 matrix. The Golden code was shown to be optimal for this system [2].

In this paper we will consider codes of length L over an alphabet $\mathcal{G}$, i.e., the transmitted codewords can be written as $\mathbf{X} = (X_1, \dots, X_t, \dots, X_L) \in \mathbb{C}^{2\times 2L}$, where $X_t \in \mathcal{G}$ is given by [2]

$$X_t = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha\left(a_t + b_t\theta\right) & \alpha\left(c_t + d_t\theta\right) \\ i\bar{\alpha}\left(c_t + d_t\bar{\theta}\right) & \bar{\alpha}\left(a_t + b_t\bar{\theta}\right) \end{bmatrix} \tag{2}$$

where $a_t, b_t, c_t, d_t \in \mathbb{Z}[i]$ are the information symbols, $\theta = 1-\bar{\theta} = \frac{1+\sqrt{5}}{2}$, $\alpha = 1+i-i\theta$, $\bar{\alpha} = 1+i(1-\bar{\theta})$, and the factor $1/\sqrt{5}$ is used to normalize energy [2]. Q-QAM constellations are used, where $Q = 2^q$ as information symbols in (2). We assume the constellation is carved from $\mathbb{Z}[i]$ and shifted in

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$1/2 + i/2$. When no outer coding is applied, we have the *uncoded Golden code*, while if a trellis outer code is used, we have a GST-TCM [1, 6].

The received signal matrix $\mathbf{Y} = (Y_1, \ldots, Y_t, \ldots, Y_L) \in \mathbb{C}^{2\times 2L}$, is given by

$$Y_t = H_t X_t + Z_t \qquad t = 1, \ldots, L \tag{3}$$

where each channel matrix $H_t \in \mathbb{C}^{2\times 2}$ is assumed to be constant for at least two channel uses. In (3), $\mathbf{Z} = (Z_1, \ldots, Z_L) \in \mathbb{C}^{2\times 2L}$ is the complex white Gaussian noise matrix with i.i.d. samples distributed as $\mathcal{N}_{\mathbb{C}}(0, N_0)$. The elements of H_t are assumed to be i.i.d. circularly symmetric Gaussian random variables $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$.

In a block fading channel, the matrices $H_t \in \mathbb{C}^{2\times 2}$ are assumed to be constant in a block of N consecutive alphabet symbols in $\mathcal{G}$ (i.e., $2N$ channel uses) and vary independently from one block to another, i.e.,

$$H_{kN+1} = \cdots = H_{(k+1)N} \quad \text{for} \quad k = 0, \ldots, L/N - 1$$

where we assume for convenience that N divides L. For $N = L$ we say we have a *slow* fading channel and for $N = 1$ a *fast* fading channel.

The block fading channel model is well suited to describe frequency selective channels that appear in indoor wireless local area networks (WLANs). These systems make use of the orthogonal frequency-division multiplexing (OFDM) technique, where each subcarrier signal is designed to go through a flat fading subchannel. The MIMO extension of these systems makes use of Space-Frequency codes, where each antenna transmits OFDM symbols [9].

Let W denote the total channel bandwidth and T_ℓ denote the maximum latency that can be tolerated by the real time applications. Let T_c be the channel *coherence time* and B_c the channel *coherence bandwidth*. The block fading model describes a channel where the coefficients are approximately constant over a frame of duration T_c and vary independently from one frame to another. Similarly, in the frequency domain, the channel transfer function is approximately constant over a subband of width B_c and varies independently from one subband to another.

Indoor wireless channels are mostly impaired by multipath, which results in a relatively small B_c. On the other hand the reduced mobility within the indoor environment results in a relatively large T_c. Using the OFDM technique we discretize the time-frequency plane (T_ℓ, W) into time-frequency slots of size $(\Delta t, \Delta f)$. Let $N_t = T_\ell/\Delta t$ denote the number of OFDM symbols that can be transmitted in a frame and $N_f = W/\Delta f$ denotes the number of subcarriers within each OFDM symbol.

In this scenario, it is common practice to design systems where $T_\ell \leq T_c$ and $\Delta f \approx B_c$, which results in a *slow* fading in time and a *fast* fading in frequency (see Figs. 1 and 2). With this choice each transmitted frame will go through a non time-varying channel with transfer function $H(f, t) = H(f)$.

Depending on the application, a coded system will employ a certain number N_s of time-frequency slots within a frame to transmit one codeword. We will assume that N_s divides

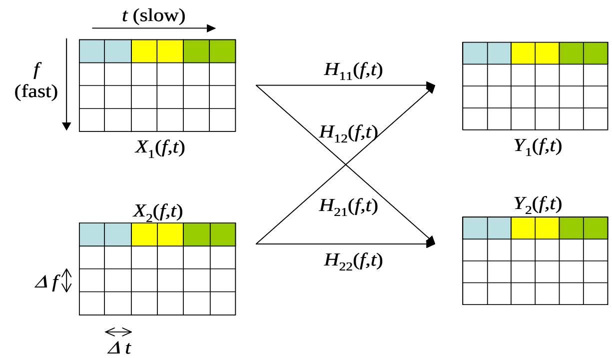

Fig. 1. Space-Time-Frequency codeword allocation in a 2×2 MIMO system: the codeword is transmitted through a slow fading channel.

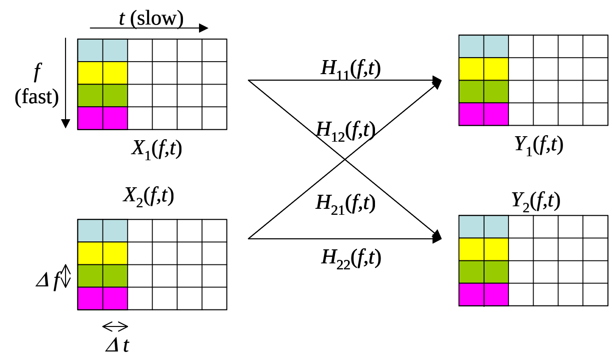

Fig. 2. Space-Time-Frequency codeword allocation in a 2×2 MIMO system: the codeword is transmitted through a fast fading channel.

exactly the total number $N_t N_f$ of time-frequency slots within a frame, i.e., $N_t N_f = K N_s$, where K is the number of codewords per frame.

In the 2×2 MIMO case, when the antenna separation is sufficiently large, we have $n_T n_R = 4$ independent channels that can be exploited to gain diversity. In order to transmit a 2×2 Golden codeword X we need $N_s = 2$ time-frequency slots. Since the Golden code is a space-time code designed for a slow fading channel in (1), we must choose two slots in consecutive OFDM symbols that have a non time-varying channel.

Assume we want to transmit codewords $\mathbf{X} = [X_1, \ldots, X_L]$ which are obtained by concatenating the Golden code with some outer code. The first row of $\mathbf{X}$ contains the time-frequency samples of the signal $X_1(f, t)$ and is sent over the first antenna. The second row of $\mathbf{X}$ contains the time-frequency samples of the signal $X_2(f, t)$ and is sent over the second antenna. We have different options for positioning the components of $\mathbf{X}$ in the time-frequency frame.

Figure 1 shows the case where $K = N_f$ codewords are sent over $2L$ consecutive time slots within the same OFDM frequency subband. If $2L\Delta t \approx T_c$ we have the slow fading channel in each subband described by the following relation

$$\mathbf{Y} = H_k \mathbf{X} + \mathbf{Z}, \qquad k = 1, \ldots, N_f \tag{4}$$

where $\mathbf{Z} \in \mathbb{C}^{2\times 2L}$ is the complex white Gaussian noise matrix.

Figure 2 shows the case where each codeword is sent within two consecutive OFDM symbols. A total of $K = N_t/2$ codewords are sent over L frequency slots. If $\Delta f \approx B_c$ we have the fast fading channel given by (3). In practice, a frequency interleaver is often inserted in order to provide better independence between the channel coefficient matrices H_k in different subbands. For convenience we will assume that $L = N_f$, nevertheless other lengths can be easily adapted to the frame if L is an integer fraction or multiple of N_f.

Other allocations of the codeword in the time-frequency plane will result in a general N-block fading channel model.

In this paper, we assume that the channel is known at the receiver. This can be obtained by sending some pilot symbols to estimate the channel at the receiver. Note that at least one reference/pilot OFDM symbol per coherence time is needed in order to track the channel variations.

III. Code Design Criteria of GST-TCM for Block Fading Channels

Assuming that a codeword X_t is transmitted, the maximum-likelihood receiver might decide erroneously in favor of another codeword $\hat{X}_t$, resulting in a *pairwise error event*. Let r denote the rank of the *codeword difference matrix* $X_t - \hat{X}_t$. Let $A_t = (X_t - \hat{X}_t)(X_t - \hat{X}_t)^\dagger$ be the *codeword distance matrix*. The pairwise error probability (PEP) depends on the determinant $\det(A_t)$ and r [7].

The union bound (UB) gives an upper bound to the performance of the STBC, while a truncated UB gives an asymptotic approximation [10]. The dominant term in the UB is the pairwise error probability (PEP) that depends on the *minimum determinant* of the codeword distance matrix

$$\Delta_{\min} = \min_{X_t \neq \hat{X}_t} \det(A_t)$$

Following the code design criteria for space-time coding in [7] that is based on the minimization of the dominant term in the UB, we call $n_T n_R$ the *diversity gain* and $(\Delta_{\min})^{1/n_T}$ the *coding gain*.

In this paper we will focus on the Golden code which is linear and full rank (i.e., $r = n_T = 2$ for all $A_t = X_t X_t^\dagger$s) and we will consider the truncated UB with two terms

$$P(e) \approx N_{s_1} P(\Delta'_1) + N_{s_2} P(\Delta'_2) \tag{5}$$

where the $P(\Delta'_i), i = 1, 2$, are the approximate PEPs of the two dominating events and N_{s_i} the corresponding multiplicities. Note that $\Delta'_{\min} = \min(\Delta'_1, \Delta'_2)$.

As we have seen, in a block fading channel H_t is constant for $2N$ channel uses. This implies that the number of blocks within a codeword experiencing independent fading channels is $B = L/N$.

In the case of linear codes, we simply consider the distance from the all-zero codeword matrix. For a given codeword $\mathbf{X}$, we define

$$F_\ell = \sum_{t=(\ell-1)N+1}^{\ell N} X_t X_t^\dagger \qquad \ell = 1, \ldots, B \tag{6}$$

We then have

$$\Delta_{\min} = \min_{\det(F_\ell) \neq 0} \prod_{\ell=1}^{B} \det(F_\ell) \tag{7}$$

A code design criterion attempting to maximize $\Delta_{\min}$ is hard to exploit, due to the non-additive nature of the determinant metric in (7). Since $X_t X_t^\dagger$ are positive definite matrices, we use the following determinant inequality [11]

$$\det(F_\ell) \geq \sum_{t=(\ell-1)N+1}^{\ell N} \det\left(X_t X_t^\dagger\right) = a_\ell \tag{8}$$

and

$$\Delta_{\min} \geq \min_{a_\ell \neq 0} \prod_{\ell=1}^{B} a_\ell = \Delta'_{\min} \tag{9}$$

In order to maximize $\Delta'_{\min}$ to design good codes, we should

- design set partitioning that maximize the a_ℓ in (8);
- design trellis codes to increase the minimum number of non zero terms a_ℓ.

This design approach is hard to realize, since we cannot easily control the a_ℓs. Instead, we focus on the design criteria of GST-TCM for slow fading in [1, 6]. We show how it can also provide a good performance for an arbitrary block fading channel. In fact, we will see that the codes designed in [1, 6] are robust to any type of block fading channel ranging from slow to fast.

IV. Performance Analysis of GST-TCM on Block Fading Channels

The aim of this section is to analyze the impact of different block fading channel conditions on the code performance of GST-TCM. This will provide the intuition about the robustness of the system when channel ranges from slow fading to fast. Although the analysis is based on several approximations, it does agree with the simulation results.

Given a GST-TCM, let S denote the length of the shortest simple error event in the corresponding trellis diagram [1]. A GST-TCM codeword spans L Golden code alphabet symbols, hence we can have $N_s = L - S + 1$ simple error events. Assuming the codeword spans $B = L/N$ independent fading blocks of length N, the simple error events will appear in different blocks depending on their position and length. We have that a simple error event is either crossing

1) $n_1 = \lfloor S/N \rfloor$ consecutive blocks
2) $n_2 = \lfloor S/N \rfloor + 1$ consecutive blocks

where $\lfloor x \rfloor$ denotes the maximum integer smaller or equal to x. The corresponding numbers of simple error events in case 1 and case 2 are respectively

$$\begin{aligned} N_{s_1} &= (B' - 1) \times r \\ N_{s_2} &= B' \times (N - r) \end{aligned}$$

where

$$B' = B - \left\lfloor \frac{S-1}{N} \right\rfloor$$

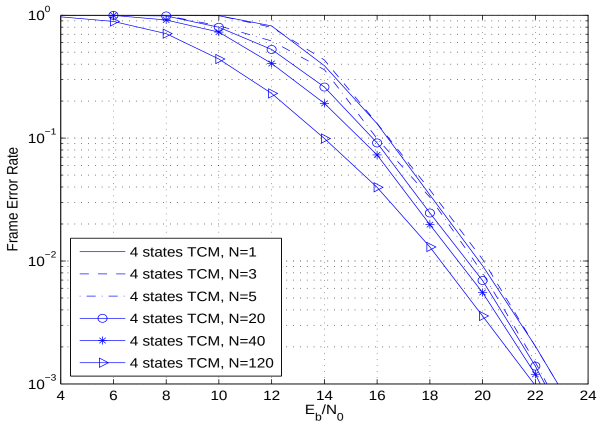

Fig. 3. Comparison of 4-state trellis codes using 16-QAM constellation at the rate 7 bpcu form a three level partition $\mathbb{Z}^8/E_8$ $(S = 2)$.

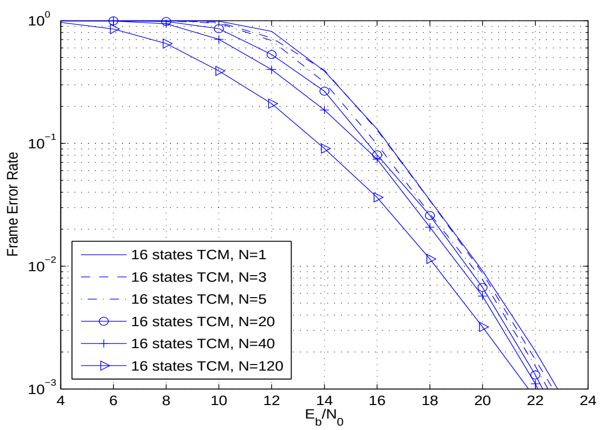

Fig. 4. Comparison of 16-state trellis codes using 16-QAM constellation at the rate 7 bpcu form a three level partition $\mathbb{Z}^8/E_8$ $(S = 3)$.

is the number of blocks not interested by edge effects,

$$r = S - \left\lfloor \frac{S-1}{N} \right\rfloor \times N - 1$$

and $N_s = N_{s_1} + N_{s_2}$.

In order to evaluate the dominant terms in (5) we look at the contribution of the simple error events in the trellis together with their multiplicity. We get N_{s_1} terms with

$$\Delta_1' = \min_{\ell} \prod_{n=0}^{n_1-1} a_{\ell+n}$$

and N_{s_2} terms with

$$\Delta_2' = \min_{\ell} \prod_{n=0}^{n_2-1} a_{\ell+n}$$

Depending on the length and structure of the simple error events, the Δ_1' and Δ_2' together with their multiplicity N_{s_1}, N_{s_2} will dominate the performance of the coding scheme.

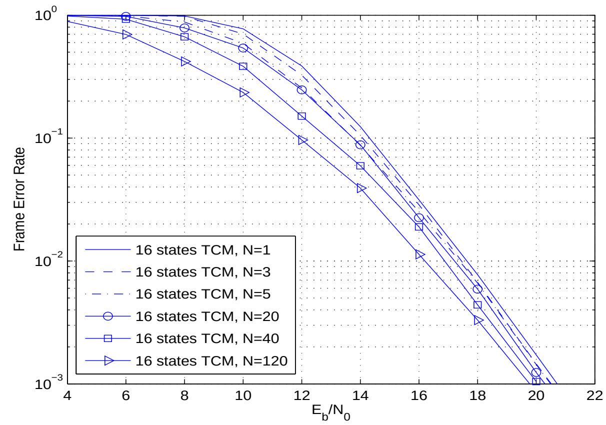

Fig. 5. Comparison of 16-state trellis codes using 16-QAM constellation at the rate 6 bpcu form a three level partition $\mathbb{Z}^8/L_8$ $(S = 3)$.

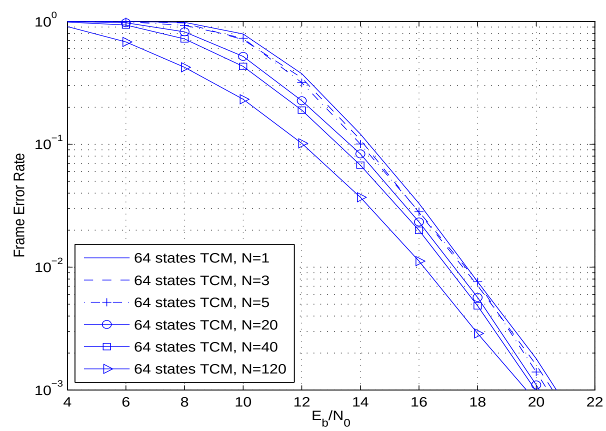

Fig. 6. Comparison of 64-state trellis codes using 16-QAM constellation at the rate 6 bpcu form a three level partition $\mathbb{Z}^8/L_8$ $(S = 4)$.

Even if we have Δ_2' smaller than Δ_1' its contribution to the overall performance can be mitigated by the fact that $N_{s_1} \gg N_{s_2}$. We will see in the following section how the a_ℓ are affected by the trellis code structure.

V. Simulation Results

In this section we show the performance of different GST-TCM schemes over block fading channels. Signal to noise ratio (SNR) is defined as $\text{SNR} = n_T E_b/N_0$, where $E_b = E_s/q$ is the energy per bit and q denotes the number of information bits per QAM symbol of energy E_s.

We consider two types of GST-TCM based on the two and three level partitions $\mathbb{Z}^8/E_8$ and $\mathbb{Z}^8/L_8$ in [1]. For each case we consider trellises with 4 or 16 states and 16 or 64 states, respectively. The length of the simple error events is $S = 2, 3, 4$ for 4,16 and 64 states, respectively. We assume the codeword length is $L = 120$ and the block fading channels are characterized by $N = 1, 3, 5, 20, 40, 120$. The GST-TCM were optimized in [1] for the slow fading channel, i.e., for $N = 120$.

S	step 1	step 2	step 3	step 4
2	δ	2δ		
3	2δ	δ	2δ	
3	4δ	δ	2δ	
4	4δ	δ	2δ	4δ

TABLE I

SEQUENCES OF $\det(X_t X_t^\dagger)$ FOR THE SIMPLE ERROR EVENTS OF THE GST-TCMS IN FIGS. 3-6 ($\delta = 1/5$).

St.	N	N_{s_1}	N_{s_2}	n_1	n_2	Δ_1'	Δ_2'
4	1	119	–	2	–	$2\delta^2$	–
4	3	80	39	1	2	3δ	$2\delta^2$
4	5	96	23	1	2	3δ	$2\delta^2$
4	20	114	5	1	2	3δ	$2\delta^2$
4	40	117	2	1	2	3δ	$2\delta^2$
4	120	119	–	1	–	3δ	–
16	1	118	–	3	–	$4\delta^3$	–
16	3	40	78	1	2	5δ	$2\delta^2 + 2\delta$
16	5	72	46	1	2	5δ	$2\delta^2 + 2\delta$
16	20	108	10	1	2	5δ	$2\delta^2 + 2\delta$
16	40	114	4	1	2	5δ	$2\delta^2 + 2\delta$
16	120	118	–	1	–	5δ	–

TABLE II

SIMPLE ERROR EVENTS FOR 4, 16 STATES $\mathbb{Z}^8/E_8$ GST-TCM, $S = 2, 3$ AND DIFFERENT BLOCK FADING CHANNELS ($N = 1, 3, 5, 20, 40, 120$).

In Figures 3-6 we can see that the best performance is obtained in the slow fading case ($N = 120$), for which the codes were explicitly optimized. The worst performance appears in the fast fading case ($N = 1$) although the difference is about 1.5-2dB a 10^{-2} and only about 1dB a 10^{-3}. The slow and fast fading curves will eventually cross since the fast fading offers a higher diversity order. The intermediate cases of block fading, between fast and slow, exhibit a degrading performance as N decreases.

Let us analyze these simulation results using the truncated UB. The sequences of $\det(X_t X_t^\dagger)$ of the simple error events of the GST-TCMs in Figs. 3-6 are given in Table I, where $\delta = 1/5$ is the minimum determinant of the Golden code.

Tables II-III show the parameters for Figs. 3-6. When $N = 1$ or $N = 120$, the term Δ_1' and its multiplicity N_{s_1} dominate the performance. We see that Δ_1' for $N = 120$ is always greater than that for $N = 1$, provided $\delta = 1/5$ and a fixed N_{s_1}. This results in a better performance when $N = 120$. The same observation can be found for 64-state GST-TCM when $N = 3$.

For the other cases, we note that Δ_2' is always smaller than Δ_1' since $\delta = 1/5$. As N increases the multiplicity N_{s_2} of the Δ_2' term decreases, while N_{s_1} of the Δ_1' term increases, which results in a better performance. All these results agree with the performance analysis.

VI. CONCLUSIONS

In this paper, we present the design criteria of GST-TCM over general block fading channels. Since the code

St.	N	N_{s_1}	N_{s_2}	n_1	n_2	Δ_1'	Δ_2'
16	1	118	–	3	–	$8\delta^3$	–
16	3	40	78	1	2	7δ	$4\delta^2 + 2\delta$
16	5	72	46	1	2	7δ	$4\delta^2 + 2\delta$
16	20	108	10	1	2	7δ	$4\delta^2 + 2\delta$
16	40	114	4	1	2	7δ	$4\delta^2 + 2\delta$
16	120	118	–	1	–	7δ	–
64	1	117	–	4	–	$32\delta^4$	–
64	3	117	–	2	–	$28\delta^2$, $40\delta^2$	–
64	5	48	69	1	2	11δ	$28\delta^2$, $40\delta^2$
64	20	102	15	1	2	11δ	$28\delta^2$, $40\delta^2$
64	40	111	6	1	2	11δ	$28\delta^2$, $40\delta^2$
64	120	117	–	1	–	11δ	–

TABLE III

SIMPLE ERROR EVENTS FOR 16, 64 STATES $\mathbb{Z}^8/L_8$ GST-TCM, $S = 3, 4$ AND DIFFERENT BLOCK FADING CHANNELS ($N = 1, 3, 5, 20, 40, 120$).

construction and optimization can be difficult to implement, we analyze the impact of the block fading channel on the code performance by using a truncated Union Bound technique. The analysis shows that the performance of the GST-TCM designed for slow fading channel varies slightly if the channel condition varies from slow to fast. It is further demonstrated by simulation that the performance degrades at most 1 dB at the FER of 10^{-3} when block fading varies from slow to fast. This robust coding scheme can be particularly beneficial for high rate WLANs transmission.

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