Geometrically Uniform Differential Vector Signaling Schemes

Ezio Biglieri
Departament TIC
Universitat Pompeu Fabra, Barcelona, Spain
Email: e.biglieri@ieee.org

Emanuele Viterbo
Department of Electrical and Computer Systems Engineering
Monash University, Melbourne, Australia
Email: emanuele.viterbo@monash.edu

***Abstract*—Using an algebraic approach, we examine the design and the performance of geometrically uniform line coding schemes transmitting b bits over $w = b + 1$ wires and obtained from a subset of a permutation modulation signal set.**

I. Introduction and motivation of the work

Transmission on parallel wireline links (as those used to interconnect integrated circuits, or a television set to a set-top box) is affected by disturbances placing a number of constraints on the design of the signaling scheme. The key problem here is the design of line codes allowing the transmission of b bits over $w \geq b$ wires and using a codebook $\mathcal{W}$ subject to constraints to be detailed later. The general scheme is shown in Fig. 1. Here, b binary information symbols ± 1 are

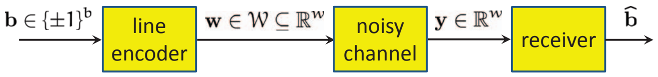

Fig. 1. *General scheme of vector coding.*

input in parallel to the (w, b) line encoder, which is a one-to-one map from $\{\pm 1\}^b$ to the codebook $\mathcal{W} \subseteq \mathbb{R}^w$. This outputs a code vector $\mathbf{w}$ with w real components, which is added to white Gaussian noise to obtain vector $\mathbf{y}$. Vector $\mathbf{y}$ is processed by a detector which outputs an estimate $\widehat{\mathbf{b}}$ of the information vector.

The single-ended $(1, 1)$ line encoding scheme, which uses one wire per link while all links share a ground signal for return current, is the simplest choice, and, with binary signaling, has a *wire efficiency* of 1 bit/wire. This scheme, which associates binary data with signal levels V and 0, wastes half of the transmitted power in a dc component, and, requiring the use of a reference level for detection, makes the system sensitive to common-mode noise sources like power supply noise and crosstalk. With $(2, 1)$ binary *differential signaling* (DS), each link needs two wires, and, since detection requires no reference level, this transmission scheme is insensitive to common-mode disturbances. In addition, the variations of power supply current are virtually zero in DS because the total power supply is kept constant (see, e.g., [1], [5], [7] for further details). However, differential signaling reduces the wire efficiency to 0.5 bit/wire. Recent work (partially listed in the References section) has focused on the design of signaling schemes that retain the advantages of DS while increasing its wire efficiency.

In this paper we take an algebraic/geometric approach to the design and analysis of line coding schemes transmitting b bits over $w = b + 1$ wires. We show how a matrix with orthogonal rows can transform a suitable subset of a Permutation Modulation (PM) vector set [9] into a signal constellation whose geometric representation in the Euclidean space is a b-dimensional hyperparallepiped, which leads to a simple decoding algorithm based on one slicer per wire. Our design accounts for several types of impairments that may be present besides additive white Gaussian noise. Specifically, resistance to common-mode noise is obtained by using code words whose components sum to zero, simultaneous switching output noise is reduced by using constant-energy signals, and the effects of intersymbol interference are reduced by having only two amplitude values at the input of each slicer. [5], [7]

Codebook design is based on the theory of Group Codes for the Gaussian Channel [10] and on the theory of groups generated by reflections in orthogonal hyperplanes. In this paper we limit ourselves to the description of a couple of examples, while the general theory will be described in a forthcoming paper [3].

II. Definitions and first properties

As mentioned before and described for example in [1], [5], [7], the constraints imposed by the transmission scheme call for the use of equal-energy code vectors $\mathbf{w}$, which are conveniently represented geometrically as points on the surface of a sphere with radius $\sqrt{\mathcal{E}}$, where $\mathcal{E}$ is the common energy of the code vectors. It is convenient to order these vectors as rows of a *codebook matrix* $\mathbf{W}$. For reasons that will be apparent from the ensuing discussion, we are interested in codebooks exhibiting a large degree of symmetry, viz., *geometrical uniformity*. This property is defined as follows (see, e.g., [4]):

Definition 1 *The codebook $\mathcal{W}$ is* geometrically uniform *(GU) if, given any two vectors $\mathbf{w}_i$, $\mathbf{w}_j$ in $\mathcal{W}$, there exists an isometry $U_{i\to j}$ that transforms $\mathbf{w}_i$ into $\mathbf{w}_j$ while leaving $\mathcal{W}$ invariant:*

$$\begin{aligned} U_{i\to j}\mathbf{w}_i &= \mathbf{w}_j \\ U_{i\to j}(\mathcal{W}) &= \mathcal{W} \end{aligned} \tag{1}$$

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We say that $\mathcal{W}$ has the *uniform distance* (UD) property if the set of distances from any codeword $\mathbf{w}$ to the other codewords is the same for all $\mathbf{w} \in \mathcal{W}$, and that it has the *uniform Voronoi* (UV) property if all the Voronoi regions of the codewords are congruent. If $\mathcal{W}$ is geometrically uniform, then it has the uniform distance property and the uniform Voronoi property. A GU codebook may be generated by applying to an *initial vector* $\mathbf{w}_1$ a set of orthogonal matrices forming a group [10]. Here we are interested in group codes that are subsets of a PM set (and hence have words that are permutations of $\mathbf{w}_1$), and in addition are represented geometrically as the vertices of a hyperparallelepiped in the Euclidean space $\mathbb{R}^b$. This can be obtained by applying to an initial vector a group of matrices generated by reflections in orthogonal hyperplanes [3].

III. Vector signaling using permutation modulation

A way of generating a GU codebook is by using PM [9]. A (Variant-I) PM codebook is obtained as the set of all the permutations of an initial n-vector $\mathbf{w}_1$. These are in number of $n!/(m_1!m_2!\cdots m_r!)$, where there are r distinct components in $\mathbf{w}_1$ with multiplicities $m_1, \dots, m_r$, and $\sum_{i=1}^r m_i = n$. The additional requirement [5] that each codeword be balanced, i.e, the sum of its components be zero, leads to a scheme which is optimum in the sense of maximizing the minimum Euclidean distance between words [2]. A peculiar feature of PM is the existence of a relatively simple optimum (ML) detection scheme over the AWGN channel. For Variant-I PM, this consists of ordering the received vector $\mathbf{y}$ in decreasing order of its entries, and choosing the transmitted vector whose order matches that of $\mathbf{y}$. Thus, the optimum receiver can be thought of as the combination of $\binom{n}{2}$ comparators (obtaining the signs of all pairwise differences between wire signal levels) followed by a lookup table. PM codebooks are GU, as they are obtained by the operation of a group of orthogonal matrices on the initial vector (for Variant-I PM, this is the symmetric group of $n \times n$ permutation matrices).

The examination of the ML detection rule of PM codebooks leads to the definition of GU codebooks admitting an especially simple detection rule, as we shall be describing in the following. For illustration's sake, we start with a simple special case. Consider the PM codebook with 6 words obtained as all the permutation of the components of the initial vector $\mathbf{w}_1 = (-1, 0, 1)$. Denoting by i-j the difference between the ith and the jth components of a vector, this codebook can be decoded using only the signs of the differences 1-2, 2-3, and 1-3 of the observed vector $\mathbf{y}$. These are necessary to order the components of $\mathbf{y}$. The situation is summarized in Table I, where those differences are shown for all the codebook vectors.

TABLE I
Vectors of a PM codebook and differences between their components.

	vector	1-2	2-3	1-3
①	$(-1,0,1)$	-1	-1	-2
②	$(-1,1,0)$	-2	$+1$	-1
③	$(0,-1,1)$	$+1$	-2	-1
④	$(1,0,-1)$	$+1$	$+1$	$+2$
⑤	$(0,1,-1)$	-1	$+2$	$+1$
⑥	$(1,-1,0)$	$+2$	-1	$+1$

Now, we can interpret the operations summarized in Table I as a linear mapping $\mathcal{L}$ between the original codebook $\mathcal{W}$ and the transformed codebook, denoted $\mathcal{W}' = \mathcal{L}\mathcal{W}$, whose entries are listed in the right part of Table I. Since this transformation is one-to-one, ML detection of $\mathcal{W}'$ is equivalent to the detection of the original codebook $\mathcal{W}$. For this observation to be practically useful, we need to consider PM schemes such that $\mathcal{W}'$ admits a very simple ML detection rule. The simplest situation, and the one on which we focus our attention here, occurs when the tips of the vectors in codebook $\mathcal{W}'$ are at the vertices of a hyperparallelepiped. If all the vertices of the hyperparallelepiped are included (which may be obtained when the number M of codewords chosen is a power of 2, say $M = 2^b$), then $\mathcal{W}'$ *can be optimally detected by simply taking the sign of each entry* of the transformed vector $\mathcal{L}\mathbf{y}$, i.e., feeding these to a slicer. Thus, the ML receiver in this situation consists of a linear transformation $\mathcal{L}$ followed by a set of b slicers. This fact can also be used for *encoding* purposes: in fact, as we shall illustrate through some examples, *even encoding can be done linearly*, by applying the inverse of $\mathcal{L}$ to any vector containing, in a suitable form, b entries of the form $(\pm1, \pm1, \dots, \pm1)$. We are now ready to illustrate the above concepts by using two design examples.

A. A design example with $b = 2$

Here we show the design of a GU codebook suitable for transmission of 2 bits over 3 wires. The codebook with $2^b = 4$ words obtained by removing two vectors from the 6-vector PM set of Table I has the matrix form

$$\mathbf{W} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{matrix} ① \\ ② \\ ④ \\ ⑥ \end{matrix} \qquad (2)$$

The words of this codebook, being balanced, can be represented in a 2-dimensional space by using the Peterson coordinate transformation [8] described by the orthogonal matrix

$$\mathbf{A} = \begin{bmatrix} 1+\beta & \beta & 1/\sqrt{3} \\ \beta & 1+\beta & 1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \qquad (3)$$

where $\beta \triangleq -1/(3-\sqrt{3})$. By using this, the codebook vectors are transformed into vectors whose third component is zero, thus reducing the codebook representation to a two-dimensional space (see Fig. 2). This codebook is a geometrically uniform signal set generated by two matrices describing

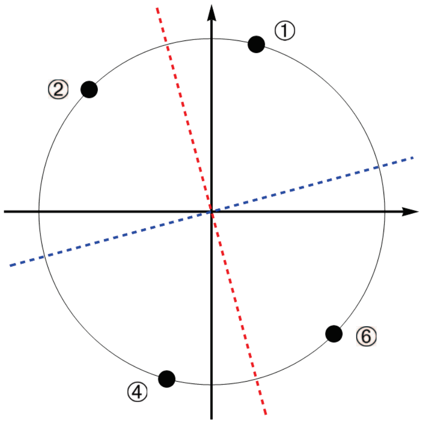

Fig. 2. *2-dimensional representation of codebook* (III-A). *Dashed lines: Separators of ML decision regions based on the signs of* $2-3$ *and of* $(1-2)+(1-3)$.

the reflections in the two orthogonal dashed lines of Fig. 2. The ML detection regions are defined by the two boundary lines described by the equations

$$\langle \mathbf{y}, (\mathbf{w}_i - \mathbf{w}_j) \rangle = 0 \qquad (4)$$

where $\langle \cdot, \cdot \rangle$ denotes scalar product and $\mathbf{w}_i$, $\mathbf{w}_j$ are neighbors. Eq. (4) expresses the fact that $\mathbf{y}$ has the same distance from $\mathbf{w}_i$ and $\mathbf{w}_j$, and hence the separating plane is orthogonal to the line joining $\mathbf{w}_i$ and $\mathbf{w}_j$.

In two dimensions, we can easily visualize the ML detection (or Voronoi) regions bounded by the line separating the neighbors $\mathbf{w}_1$ and $\mathbf{w}_2$, which in 3 dimensions has equation $y_2 - y_3 = 0$, and by the plane separating the neighbors $\mathbf{w}_1$ and $\mathbf{w}_6$, which in 3 dimensions has equation $2y_1 - y_2 - y_3 = 0$. This calculation shows that the codebook (2) can be optimally decoded using the signs of $(y_2 - y_3)$ and of $(2y_1 - y_2 - y_3)$, or, equivalently, of $(y_1 - y_2) + (y_1 - y_3)$ (see Table II).

TABLE II
Vectors of a subset of the PM codebook of Table I and their transformed version.

	vector	$(1-2)+(1-3)$	$2-3$
①	$(-1, 0, 1)$	3	-1
②	$(-1, 1, 0)$	-3	1
④	$(1, 0, -1)$	3	1
⑥	$(1, -1, 0)$	-3	-1

We can interpret the detection summarized in Table II as describing the transformation $\mathcal{L}$ which maps the GU codebook $\mathcal{W}$, generated by the action of a 4-element group of orthogonal matrices, onto the GU vector set $\mathcal{W}'$ with 4 elements $\pm 3, \pm 1$. The latter vector set can be represented geometrically by the four vertices of a rectangle. The decision regions of the transformed codebook are the coordinate axes in the 2-dimensional plane, and hence the transmitted vector can be detected by simply slicing the components of $\mathcal{L}\mathbf{y}$, as indicated above.

The operation summarized in Table II can be viewed as the linear transformation of $\mathbf{W}$ through the *detection matrix*

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \qquad (5)$$

which has orthogonal rows (notice that $\mathbf{M}$ is not orthogonal, so that trasformation by $\mathbf{M}$ alters the scales of the coordinate axes). Its first row reflects the fact that the sum of the components of each row of $\mathbf{W}$ is zero (balanced codewords), the second row corresponds to the difference $(1-2)+(1-3)$, and the third row to the difference $(2-3)$. Thus, we have

$$\mathbf{W}\mathbf{M}^{\mathsf{T}} = \begin{bmatrix} 0 & -3 & -1 \\ 0 & -3 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & -1 \end{bmatrix} \qquad (6)$$

which expresses the transformation of the original vector set $\mathbf{W}$ into the new vector set whose two-dimensional representation is shown in Fig. 3. In turn, this vector set can be detected using simply the signs of its second and third components. Conversely, the linear coding procedure transforms the *information matrix*

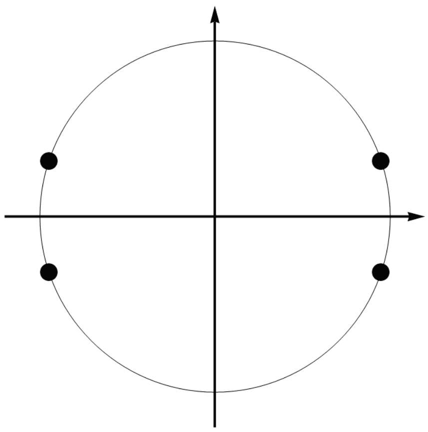

Fig. 3. *2-dimensional representation of the linearly transformed codebook* $\mathbf{W}\mathbf{M}^{\mathsf{T}}$.

$$\mathbf{B} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad (7)$$

into $\mathbf{W}$ as

$$\mathbf{W} = \mathbf{B}\mathbf{K} \qquad (8)$$

where $\mathbf{K} = \frac{1}{2}\mathbf{M}$. Notice that $\mathcal{L}$ transforms the 3-dimensional codebook $\mathbf{W}$ into the 2-dimensional codebook $\mathbf{W}'$. Thus, the inverse transformation mapping the information symbols into $\mathbf{W}$ should be a map between a 2-dimensional and a 3-dimensional space. This explains why each row of $\mathbf{B}$ has a "zero" prepended (notice also that $\mathbf{B} = \text{sgn}(\mathbf{W}\mathbf{M}^{\mathsf{T}})$).

Remark 1 Observe that the design procedure described above may not work if the codebook is not GU. To see this, consider the codebook

$$\mathbf{W} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \tag{9}$$

This is subset of the PM codebook in (2) but is not GU. Its geometrical representation is shown in Fig. 4. It is seen that

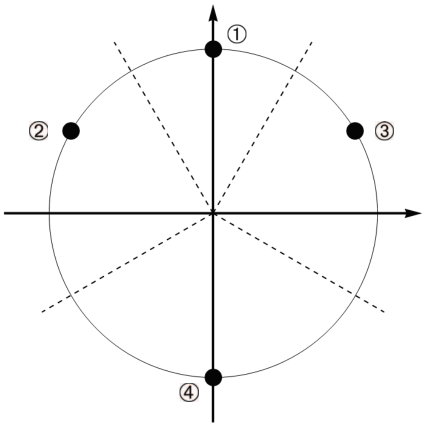

Fig. 4. 2-*dimensional representation of codebook* (9). *Dashed lines: Separators of ML decision regions.*

the ML detection regions are not congruent, and the simple detection rule valid for previous scheme is not ML here.

B. A design example with $b = 3$

This codebook (called ENRZ in [5]) has $b = 3$ and $w = 4$, and is obtained by applying to the initial vector $\mathbf{w}_1 = (-3, 1, 1, 1)$ the group of matrices generated by the following reflections in orthogonal hyperplanes:

$$\mathbf{O}_1 = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$
$$\mathbf{O}_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \tag{10}$$
$$\mathbf{O}_3 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

Application of the Peterson transformation [8] to this codebook yields a codebook matrix whose 8 rows have the form $(\pm 2, \pm 2, \pm 2, 0)$, representing a 3-dimensional cube. A simple calculation shows that the separating hyperplanes for the ML decision region of signal $(-3, 1, 1, 1)$ have equations $\langle \mathbf{y}, (-2, -2, 2, 2) \rangle = 0$, $\langle \mathbf{y}, (-2, 2, -2, 2) \rangle = 0$, and $\langle \mathbf{y}, (-2, 2, 2, -2) \rangle = 0$, which yields a decoding matrix $\mathbf{M}$ in the form of a 4×4 Hadamard matrix.

C. Error probabilities

After addition of white Gaussian noise samples $\sim \mathcal{N}(0, N_0/2)$ independent across wires and transmitted b-tuples, the codebook matrix $\mathbf{W}+\mathbf{N}$ is received. The detection process is summarized as the calculation of the signs of $(\mathbf{W}+\mathbf{N})\mathbf{M}^\mathsf{T}$. The jth symbol of the ith source b-tuple is erroneously detected if its polarity is altered by noise, which occurs with probability

$$(p_\mathrm{e})_{i,j} = \mathbb{P}\left(n_j < -\left|\left(\mathbf{W}\mathbf{M}^\mathsf{T}\right)_{i,j}\right|\right) \tag{11}$$

where $n_j \sim \mathcal{N}(0, \sigma_j^2)$, and $\sigma_j^2 \triangleq (N_0/2)\mu_j^2$ is the jth element of the diagonal covariance matrix of the noise term:

$$\begin{aligned} \mathbb{E}\left[(\mathbf{N}\mathbf{M}^\mathsf{T})^\mathsf{T}(\mathbf{N}\mathbf{M}^\mathsf{T})\right] &= \mathbf{M}\left[\mathbb{E}(\mathbf{N}^\mathsf{T}\mathbf{N})\right]\mathbf{M}^\mathsf{T} \\ &= \frac{N_0}{2}\mathrm{diag}(\mu_1^2, \ldots, \mu_{b+1}^2) \end{aligned} \tag{12}$$

Thus,

$$(p_\mathrm{e})_{i,j} = \mathrm{Q}\left(\frac{\left|\left(\mathbf{W}\mathbf{M}^\mathsf{T}\right)_{i,j}\right|}{\sqrt{N_0/2}\mu_j}\right) \tag{13}$$

with $\mathrm{Q}(\,\cdot\,)$ the Gaussian tail function. We define the signal-to-noise ratio η observing that the average energy associated with the transmission of a signal b-tuple is given by

$$\mathcal{E} = \frac{\|\mathbf{W}\|^2}{2^b} \tag{14}$$

where $\|\mathbf{W}\|$ denotes the Frobenius norm of matrix $\mathbf{W}$. The energy per bit is consequently $\mathcal{E}_b = \mathcal{E}/b$, and the signal-to-noise ratio is

$$\eta \triangleq \frac{\mathcal{E}_b}{N_0} = \frac{\|\mathbf{W}\|^2/2^b}{bN_0} \tag{15}$$

Thus, we can rewrite (13) in the form

$$(p_\mathrm{e})_{i,j} = \mathrm{Q}\left(\alpha_{i,j}\sqrt{2\eta}\right),\ i = 1, \ldots, 2^b,\ j = 2, \ldots, b+1 \tag{16}$$

where

$$\alpha_{i,j} \triangleq \frac{\left(\left|\mathbf{W}\mathbf{M}^\mathsf{T}\right|(\mathbf{M}\mathbf{M}^\mathsf{T})^{-1/2}\right)_{i,j}}{\sqrt{\|\mathbf{W}\|^2/(b2^b)}} \tag{17}$$

Observe that the matrix $\left|\mathbf{W}\mathbf{M}^\mathsf{T}\right|$ quantifies the amplitudes of the eye opening before rectification.

If we define the $2^b \times b$ matrix $\boldsymbol{\alpha}$ whose entries are $\alpha_{i,j}$, $i = 1, \ldots, 2^b$, $j = 2, \ldots, b+1$, we may define a matrix with entries (16) and containing all the information needed to derive error probability bounds, viz.,

$$\mathbf{P}(e) \triangleq \mathrm{Q}\left(\sqrt{2\eta}\boldsymbol{\alpha}\right) \tag{18}$$

Due to the geometric uniformity of the codebook, $\boldsymbol{\alpha}$ has equal rows, indicating that all signal vectors are equally sensitive to the effects of noise. The sum of the entries of $\mathbf{P}(e)$ in any row is the upper union bound to the conditional error probability given that the ith information b-tuple is transmitted. Notice also that equal entries in any row indicate that all symbols of a source vector are equally sensitive to noise.

Example 1 Binary differential signaling is a special case of orthogonal vector signaling. This $(2, 1)$ line code has

$$\mathbf{W} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{19}$$

which can be generated as the product $\mathbf{BK}$, where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \tag{20}$$

and $\mathbf{K}$ is the Hadamard matrix

$$\mathbf{K} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{21}$$

Choosing $\mathbf{M} = \mathbf{K}$, we have

$$\mathbf{WM}^{\mathsf{T}} = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} \tag{22}$$

so that $\operatorname{sgn}(\mathbf{WM}^{\mathsf{T}}) = \mathbf{B}$, as it should be. Error probability can be evaluated by direct computation of

$$\boldsymbol{\alpha} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{23}$$

which yields

$$\mathbf{P}(e) = \begin{bmatrix} \mathrm{Q}\big(\sqrt{2\eta}\big) \\ \mathrm{Q}\big(\sqrt{2\eta}\big) \end{bmatrix} \tag{24}$$

■

Example 2 For the codebook (2), $\mathbf{WM}^{\mathsf{T}}$ is as in (6), and hence $\operatorname{sgn}(\mathbf{WM}^{\mathsf{T}}) = \mathbf{B}$, as it should be. Encoding uses the 3×3 coding matrix $\mathbf{M}$ of (5), which yields, after choosing $\varepsilon = 1/2$ to obtain values in the interval $[-1, 1]$:

$$\frac{1}{2}\mathbf{BK} = \mathbf{W} \tag{25}$$

$\mathbf{M}$ and $\mathbf{K}$ have the property that the product $\mathbf{D} \triangleq \mathbf{KM}^{\mathsf{T}}$ is the diagonal matrix $\operatorname{diag}(3, 6, 2)$, and hence $\mathbf{MM}^{\mathsf{T}} = \operatorname{diag}(3, 6, 2)$. For error probability, we compute

$$\left|\mathbf{WM}^{\mathsf{T}}\right| = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \tag{26}$$

and $\|\mathbf{W}\| = 2\sqrt{2}$. Thus, the matrix $\boldsymbol{\alpha}$ has all its rows equal to $(\sqrt{3/2}\ ,\ 1/\sqrt{2})$, which shows that the four source 2-tuples have their second symbol less protected from noise than the first one, as also predicted by the shape of the transformed codebook of Fig. 3. ■

Example 3 The codebook of Subsection III-B has the rows of matrix $\left|\mathbf{WM}^{\mathsf{T}}\right|$ equal to $(0, 4, 4, 4)$, $\mathbf{MM}^{\mathsf{T}} = \operatorname{diag}(0, 4, 4, 4)$, and the rows of $\boldsymbol{\alpha}$ equal to $(1.0, 1.0, 1.0)$, indicating equal protection from noise of all source symbols.

IV. Conclusions

Elaborating on ideas introduced in [5]–[7], we have developed some concepts leading to the design of geometrically uniform line codes suitable for the transmission of b bits over $w = b + 1$ wires. We have also shown how error probabilities can be approximated.

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