Computing the Voronoi cell of a lattice: The diamond-cutting algorithm

Emanuele Viterbo and Ezio Biglieri[1]

Dipartimento di Elettronica • Politecnico • Corso Duca degli Abruzzi 24 • I-10129 Torino (Italy)

fax: +39 11 5644099 • e-mail: <name>@polito.it

Abstract — **A computational algorithm is described for the numerical evaluation of some lattice parameters such as density, thickness, dimensionless second moment (or quantizing constant), etc. By using this algorithm, previously unknown quantizing constants of some interesting lattices can be obtained.**

I. Introduction

The complete geometric structure of a lattice can be found from the description of its Voronoi cell. The knowledge of the Voronoi cell solves at once the problem of the computation of relevant lattice parameters such as packing radius, covering radius, kissing number, center density, thickness, normalized second moment (or quantizing constant).

The Voronoi cell of certain highly symmetric lattices can be determined analytically but no such result is available for an arbitrary lattice. In this paper we propose an algorithm which exactly computes the Voronoi cell of a full-rank arbitrary lattice. The exact knowledge of the Voronoi cell (i.e., knowledge of the coordinates of its vertices, edges, etc.) enables one to compute all the lattice parameters within any degree of accuracy.

The Voronoi cell of lattice is an **0**-symmetric convex polytope, i.e., a bounded region delimited by a finite number of hyperplanes symmetric about the origin. The basic elements of a polytope $\mathcal{P}$ are its *k-faces*, where k is the dimension. The 0-faces are called *vertices of* $\mathcal{P}$, the 1-faces, *edges of* $\mathcal{P}$ and the $(d-1)$-faces, *facets of* $\mathcal{P}$. For convenience we identify $\mathcal{P}$ with the d-face and the empty set with the (-1)-face. To give a complete description of a polytope we must know all the relations among its faces. For $-1 \leq k \leq d-1$ a k-face f and a $(k+1)$-face g are *incident upon* each other if f belongs to the boundary of g; in this case, f is called a *subface of* g and g a *superface of* f. The d-face represents the whole polytope and is the only superface of all the facets. The (-1)-face has no subfaces and is the only subface of all the vertices. The *incidence graph* $I(\mathcal{P})$ of $\mathcal{P}$ is an undirected graph defined as follows: for each k-face ($k = -1, 0, 1, \ldots d$) of $\mathcal{P}$, $I(\mathcal{P})$ has a node $\nu(f)$; if f and g are incident upon each other then $\nu(f)$ and $\nu(g)$ are connected by an arc.

II. The diamond-cutting algorithm

This algorithm computes the incidence graph of the Voronoi region $\mathcal{V}$ of a lattice. Its name was chosen due to its resemblance to the procedure for cutting a raw diamond into a brilliant. Let us consider a lattice Λ defined by an arbitrary basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$. Given a point $\mathbf{p}$ we will denote with $h(\mathbf{p})$ the hyperplane passing through the point $\mathbf{p}$ and normal to the vector $\mathbf{p}$. The distance of $h(\mathbf{p})$ from the origin is equal to $\|\mathbf{p}\|$.

Preparation Given the lattice basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ construct the parallelotope $\mathcal{Q}$ defined by the hyperplanes $h(\pm\frac{1}{2}\mathbf{v}_i)$ for $i = 1, \ldots, d$. $\mathcal{Q}$ contains the Voronoi cell. The corresponding incidence graph $I(\mathcal{Q})$ has 3^d nodes. Finally, set $\mathcal{V} := \mathcal{Q}$.

Cutting Consider all hyperplanes $h(\frac{\lambda_1}{2}\mathbf{v}_1 + \frac{\lambda_2}{2}\mathbf{v}_2 + \cdots \frac{\lambda_d}{2}\mathbf{v}_d)$, with λ_i integers, which cut $\mathcal{V}$ and update $I(\mathcal{V})$, by introducing the nodes corresponding to the new faces and erasing those corresponding to the faces which are left out. For this operation we have adapted Edelsbrunner's algorithm for the incrementation of arrangements [2].

Finish Compute $\mathrm{vol}(\mathcal{V})$, the volume of $\mathcal{V}$. If $\mathrm{vol}(\mathcal{V}) > \det(\Lambda)^{1/2}$ keep on cutting, else end the algorithm and output the incidence graph $\mathcal{I}(\mathcal{V})$.

III. Results

By introducing some additional information into the nodes of the incidence graph, it is possible to compute all the lattice parameters once the Voronoi cell is found. In particular we easily find the *packing radius*, the *kissing number*, the *covering radius* and the related parameters. Finding the *quantization constant* requires a slightly more complex procedure which recursively computes the volume and second order moment of $\mathcal{V}$ about **0** in terms of the volume and of the second-order moment of the subfaces.

Using the diamond-cutting algorithm we have computed some previously unknown values of the quantizing constants for some particularly interesting lattices. Of special interest are the previously unknown quantizing constants for the two locally optimal lattice coverings in $\mathbf{R}^4$ found by Dickson (Di_{4a} : 0.076993; Di_{4b} : 0.077465) and for a 5-dimensional extreme lattice covering, which belongs to the class introduced by Barnes and Trenerry (0.076278). As these lattices do not improve upon the best known lattice quantizers, the conjecture about the optimal lattice quantizers being the duals of the densest lattices still holds.

Most of the computational problems related to lattices are either known or conjectured to be NP-hard [1, p. 40]. The principal limitation in the application of the diamond-cutting algorithm is the exponentially increasing memory requirement. The possibility of reducing the memory requirements appears remote especially if we want to preserve the generality of the algorithm.

[1] This research was sponsored by the Italian National Research Council (CNR) under "Progetto Finalizzato Trasporti."

References

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