High Diversity Lattices for Fading Channels

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Abstract — **We show that particular versions of the densest lattice packings present very good performance over the Rayleigh fading channel. These versions not only have high diversity, but also may be decoded efficiently since they are binary lattices.**

I. Introduction

The practical interest in lattice constellations presenting good performance over fading channels rises from the need to transmit information at high bit rates over terrestrial radiomobile links. Constellations matched to the fading channel are effective because of their high degree of *diversity*. By diversity we intend the number of different component values of any two distinct points in the constellation. The signal constellations for Gaussian channels are usually very bad when used over Rayleigh fading channels since they have small diversity. We constructed signal constellations with high spectral efficiency matched to the Rayleigh fading channel using algebraic number theory [1]. The signal constellations are derived from the densest lattices (D_4, E_6, E_8, K_{12}, Λ_{16}, Λ_{24}) and their diversity order is half the lattice dimension.

II. System model

Consider the following model. A mapper associates an m-uple of input bits to a signal point $\mathbf{x} = (x_1, x_2, \ldots x_n)$ in the n-dimensional Euclidean space $\mathbf{R}^n$. Let $M = 2^m$ be the total number of points in the constellation. The points are transmitted over a Rayleigh channel giving $\mathbf{r} = \alpha * \mathbf{x} + \mathbf{n}$, where $\mathbf{r}$ is the received point. $\mathbf{n} = (n_1, n_2, \ldots n_n)$ is a noise vector, whose real components n_i are zero mean, N_0 variance Gaussian distributed independent random variables. $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$ are the independent random fading coefficients with unit second moment and $*$ represents the componentwise product.

The signal points $\mathbf{x}$ are chosen from a constellation which is carved from a lattice Λ. The spectral efficiency is measured in number of bits per two dimensions $s = 2m/n$, and the signal-to-noise ratio per bit is given by $SNR = E_b/N_0$, where E_b is the narrow band average energy per bit and $N_0/2$ is the narrow band noise power spectral density.

III. New constellations

An accurate analysis of the symbol error probability shows that the most important feature of a good constellation for the fading channel is its diversity L. The following theorem enables us to evaluate the diversity L of any lattice constructed from an algebraic number field.

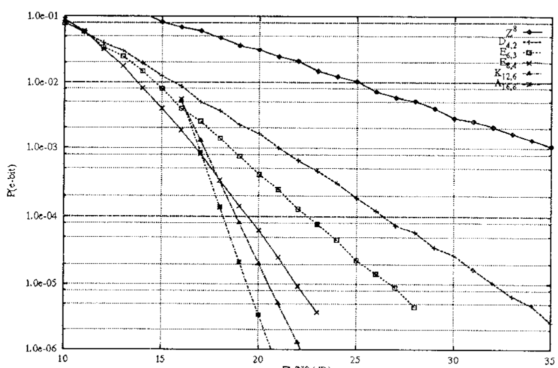
Theorem. *The lattices obtained from the canonical embedding of an algebraic number field with signature* (r_1, r_2) *exhibit a diversity* $L = r_1 + r_2$.

Since totally complex cyclotomic fields have a signature $(0, n/2)$ the diversity of the corresponding lattices is $L = n/2$. We use Craig's work [2, 3], who showed how to construct the lattices E_6, E_8, Λ_{24} (Leech lattice) from the totally complex cyclotomic fields $K = \mathbf{Q}(e^{i2\pi/N})$ for $N = 9, 20, 39$. We applied the same procedure and we found D_4 (Schlafli lattice), K_{12} (Coxeter-Todd's lattice) and Λ_{16} (Barnes-Wall's lattice) from the $8th, 21st$ and the $40th$ root of unity. These lattices are obtained by applying the canonical embedding to particular integral ideals of the above cyclotomic fields. The ideals are given in the table below. The lattices are indicated with $\Lambda_{n,L}$. Two generators for each ideal are given in the last column.

Lattice	N	Ideal
$D_{4,2}$	8	$(2, \theta + 1)$
$E_{6,3}$	9	$(3, (\theta+1)^2)$
$E_{8,4}$	20	$(5, \theta - 2)$
$K_{12,6}$	21	$(7, \theta + 3)$
$\Lambda_{16,8}$	40	$(2, \theta^4 + \theta^3 + \theta^2 + \theta + 1)(5, \theta^2 + 2)$
$\Lambda_{24,12}$	39	$(3, \theta^3 + \theta^2 - 1)(13, \theta - 3)$ $(3, \theta^3 + \theta^2 + \theta + 1)$

IV. Results

The figure below shows the performance over the Rayleigh fading channel of the rotated versions of the lattices D_4, E_6, E_8, K_{12} and Λ_{16}. Simulations were made up to dimension 8, while for higher dimensions we have plotted upper bounds. The bit error probability is given as a function of E_b/N_0 for $s = 4$ bits/symbol. The slopes of the curves asympotically correspond to the diversity order which is 2, 3, 4, 6 and 8 respectively. At 10^{-3} the gain over Z^8 is about 17dB and it exceeds 25dB at 10^{-5}. It is important to notice that



the maximal diversity reached with a reasonable trellis coded modulation does not exceed 6. The diversity of the rotated Leech lattice $\Lambda_{24,12}$ is 12. This is equivalent to a trellis coded QAM with 2^{44} states or a trellis coded PAM with 2^{22} states at 4 bits per symbol.

References

[1] J. H. Conway, N. J. Sloane: *Sphere packings, lattices and groups*, 2nd ed., 1993, Springer-Verlag, New York.

[2] M. Craig: "Extreme forms and cyclotomy," *Mathematika*, vol. 25, pp. 44-56, 1978.

[3] M. Craig: "A cyclotomic construction for Leech's lattice," *Mathematika*, vol. 25, pp. 236-241, 1978.