Algebraic 3×3, 4×4 and 6×6 Space-Time Codes with non-vanishing Determinants

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Abstract

In this paper we present algebraic constructions of 3×3, 4×4 and 6×6 Space-Time Codes, achieving full rate and full diversity. These codes have non-vanishing (in fact fixed) minimum determinants when the rate goes to infinity. Their construction is based on cyclic algebras with center equal to an algebraic field based on cyclotomic fields .

1. Introduction

In order to achieve very high spectral efficiency over wireless channels, we need multiple antennas at both transmitter and receiver ends. We are interested, here, in the coherent case where the receiver perfectly knows channel coefficients. The received signal is

$$\mathbf{Y}_{N\times T} = \mathbf{H}_{N\times M} \cdot \mathbf{X}_{M\times T} + \mathbf{W}_{N\times T} \tag{1}$$

where $\mathbf{X}$ is the transmitted codeword taken from Space-Time Block Code (STBC), $\mathbf{H}$ is the Rayleigh fading channel response and $\mathbf{W}$ is the i.i.d Gaussian noise.

We will denote by $\mathcal{C}_\infty$ [1] the infinite code where the information symbols are taken from $\mathbb{Z}[i]$ or from $\mathbb{Z}[j]$, and by $\mathcal{C}$ the finite code obtained by restricting the information symbols to q-QAM constellations ($\mathbb{Z}[i]$), with in phase and quadrature components equal to $\pm 1, \pm 3 \ldots$., or q-HEX constellations [2] ($\mathbb{Z}[j]$).

Linear dispersion Space-Time Codes (LD-STBC) have been introduced in [3]. The linearity property of the LD-STBC enables the use of ML sphere decoding, which exploits the full performance of the code compared to other suboptimal decoders [4].

Unfortunately, the structure of the LD-STBC is too “light” to construct space-time codes with the given properties.

In [5], it is shown how to construct full rate and fully diverse codes for the 2 transmit antennas case. This approach was generalized for any number of transmit antennas M in [6, 7]. All these above constructions satisfy the rank criterion and attempt to maximize the coding advantage, which is defined for LD-STBC by the *minimum determinant* of code .

We define the *minimum determinant* of $\mathcal{C}_\infty$ as

$$\delta_{\min}(\mathcal{C}_\infty) = \min_{X\in\mathcal{C}_\infty, X\neq 0} |\det(X)|^2$$

and the *minimum determinant* of $\mathcal{C}$ as

$$\begin{aligned}\delta_{\min}(\mathcal{C}) &= \min_{X_1,X_2\in\mathcal{C}, X_1\neq X_2} |\det(X_1 - X_2)|^2 \\ &\geq 2^M \delta_{\min}(\mathcal{C}_\infty)\end{aligned}$$

In [8], the authors proposed non-full-rate and full-rate STBC constructed using division algebras. A division algebra naturally yields a structured set of invertible matrices that can be used to construct LD codes (since for any codeword $\mathbf{X}\in\mathcal{C}$ the rank criterion is satisfied as $|\det(X)|^2 \neq 0$).

In all these previous constructions [6, 7, 8], the minimum determinants are non-zero, but vanish when the constellation size increases. This problem appears because transcendental elements or algebraic elements with a too higher degree are used to construct the division algebras. Non-vanishing determinants may be of interest, whenever we want to apply some outer block coded modulation scheme, which usually entails a signal set expansion, if the spectral efficiency has to be preserved. In order to obtain *energy efficient* codes we need to construct rotated versions of the complex lattices $\mathbb{Z}[i]^M$ or $\mathbb{Z}[j]^M$, so that there is no shaping loss in the signal constellation.

In paper [1], we have presented a construction of a 2×2 STBC, Golden code, having a constant minimum determinant when the constellation size increases. A code isomorphic to the Golden code was found independently in [9] and [10]. Using the same algebraic method, we present in this paper constructions of 3×3, 4×4 and 6×6 STBCs having a constant minimum determinant when the spectral efficiency increases.

2. 4×4 STBC construction

$$R_4' = \begin{pmatrix} 0.2582-0.3122i & 0.3455-0.4178i & -0.4178+0.5051i & -0.2136+0.2582i \\ 0.2582+0.0873i & 0.4718+0.1596i & 0.1596+0.054i & 0.7633+0.2582i \\ 0.2582+0.2136i & -0.5051-0.4178i & -0.4178-0.3455i & 0.3122+0.2582i \\ 0.2582-0.7633i & -0.054+0.1596i & 0.1596-0.4718i & -0.0873+0.2582i \end{pmatrix} \tag{2}$$

We will begin by the construction of the 4×4 STBC, as it is very close to that of the 2×2 STBC-Golden Code [1], where the field extension is made over $\mathbb{Q}(i)$.

Let $\mathbb{Q}(2\cos(\frac{2\pi}{n}))$ be an extension of $\mathbb{Q}$ of degree 4, which is the real-subfield of the cyclotomic field $\mathbb{Q}(\exp(i\frac{2\pi}{n}))$, n is such that $\frac{\varphi(n)}{2} = 4$, $\varphi(\cdot)$ is the totient Euler function. We will consider now the compositum field $\mathbb{K} = \mathbb{Q}(i, 2\cos(\frac{2\pi}{n}))$.

$$\begin{tikzcd}
& \mathbb{Q}(i, 2\cos(\frac{2\pi}{15})) & \\
\mathbb{Q}(i) \arrow[ur] & & \mathbb{Q}(2\cos(\frac{2\pi}{15})) \arrow[ul] \\
& \mathbb{Q} \arrow[ul, "2"] \arrow[ur, "4"'] \arrow[uu, "8"] &
\end{tikzcd}$$

Let $n = 15$, and $\mathbb{K} = \left\{a + b\theta + c\theta^2 + d\theta^3 \mid a,b,c,d \in \mathbb{Q}(i)\right\}$, as a relative extension of $\mathbb{Q}(i)$ with degree 4. Let $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[i][\theta]$ be the ring of integers of $\mathbb{K}$, with integral basis $\mathcal{B}_K = \{1, \theta, \theta^2, \theta^3\}$. We recall that for any algebraic integer $z = a + b\theta + c\theta^2 + d\theta^3 \in \mathcal{O}_{\mathbb{K}}$ with $a, b, c, d \in \mathbb{Z}[i]$. We will denote by $N_{\mathbb{K}/\mathbb{Q}(i)}(z)$ and $\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}(i)}(z)$ respectively the relative norm and the relative trace of $z \in \mathbb{K}$.

Let $\mathbb{L} = \{a + bi + c\theta + di\theta + e\theta^2 + fi\theta^2 + g\theta^3 + hi\theta^3 | a, b, \ldots, h \in \mathbb{Q}\}$ be the corresponding absolute extension of $\mathbb{K}$ over $\mathbb{Q}$, with signature $(r_1, r_2) = (0, 4)$, ring of integers $\mathcal{O}_{\mathbb{L}}$ and integral basis $\mathcal{B}_{\mathbb{L}} = \{1, i, \theta, i\theta, \theta^2, i\theta^2, \theta^3, i\theta^3\}$.[1] The relative discriminant of $\mathbb{K}$ is $d_{\mathbb{K}} = 1125 = 3^2 \cdot 5^3$, while the absolute discriminant of $\mathbb{L}$ is $d_{\mathbb{L}} = 2^8 \cdot 3^4 \cdot 5^6$.

In order to obtain energy efficient codes we need to construct a complex lattice $R\mathbb{Z}[i]^4$, where R is a complex unitary matrix, so that there is no shaping loss in the signal constellation. This lattice derives as an algebraic lattice from an appropriate relative ideal of the ring of integers $\mathcal{O}_{\mathbb{K}}$. The complex lattice $R\mathbb{Z}[i]^4$ can be equivalently seen as a rotated $\mathbb{Z}^8$-lattice: $O\mathbb{Z}^8$, O being an orthogonal matrix, obtained from an ideal of $\mathcal{O}_{\mathbb{L}}$.

A necessary condition to obtain $O\mathbb{Z}^8$ is that there exists an ideal $\mathcal{I}_{\mathbb{L}} \subseteq \mathcal{O}_{\mathbb{L}}$ with norm $45 = 3^2 \cdot 5$. In fact, the lattice $\Lambda(\mathcal{O}_{\mathbb{L}})$ has fundamental volume equals to $2^{-r_2}\sqrt{d_{\mathbb{L}}} = 1125$ and the sublattice $\Lambda(\mathcal{I}_{\mathbb{L}})$ has fundamental volume equals to $2^{-r_2}\sqrt{d_{\mathbb{L}}}N(\mathcal{I}_{\mathbb{L}}) = 3^4 \cdot 5^4 = \sqrt{15}^8$, where the norm of the ideal $N(\mathcal{I}_{\mathbb{L}})$ is equal to the sublattice index. This suggests that the fundamental parallelotope of the algebraic lattice $\Lambda(\mathcal{I}_{\mathbb{L}})$ could be a hypercube of edge length equal to $\sqrt{15}$, but this needs to be checked explicitly.

An ideal $\mathcal{I}_{\mathbb{L}}$ of norm 45 can be found from the following ideal factorizations

$$\begin{aligned} (3)\mathcal{O}_{\mathbb{L}} &= \mathcal{I}_3^2 \overline{\mathcal{I}_3}^2 \\ (5)\mathcal{O}_{\mathbb{L}} &= \mathcal{I}_5^4 \overline{\mathcal{I}_5}^4 \end{aligned}$$

Let us consider $\mathcal{I}_{\mathbb{L}} = \mathcal{I}_3 \cdot \mathcal{I}_5$. It is a principal ideal $\mathcal{I}_{\mathbb{L}} = (\alpha)$ generated by $\alpha = -1 - i - 3i\theta + i\theta^2 + i\theta^3$.

We will now define the generator matrix of $\Lambda(\mathcal{I}_{\mathbb{K}})$. Let the complex canonical embedding of $\mathbb{K}$ be defined by

$$\begin{aligned} &\sigma : \mathbb{K} \to \mathbb{C}^4 \\ &\sigma : x \mapsto (\sigma_1(x), \sigma_2(x), \sigma_3(x), \sigma_4(x)) \end{aligned} \tag{3}$$

where

$$\sigma_1(\theta) = \theta \quad \sigma_2(\theta) = i\theta \quad \sigma_3(\theta) = -\theta \quad \sigma_4(\theta) = -i\theta \tag{4}$$

The relative basis of $\mathcal{I}_{\mathbb{K}}$ is $\mathcal{B}_{\mathcal{I}_{\mathbb{K}}} = \left\{\alpha, \alpha\theta, \alpha\theta^2, \alpha\theta^3\right\}$. By applying the canonical embedding σ to $\mathcal{B}_{\mathcal{I}_{\mathbb{K}}}$ we obtain the generator matrix of $\Lambda(\mathcal{I}_{\mathbb{K}})$, which we normalize by $\frac{1}{\sqrt{15}}$.

$$R = \frac{1}{\sqrt{15}} \cdot \begin{bmatrix} \sigma_1(\alpha) & \sigma_1(\alpha\theta) & \sigma_1\left(\alpha\theta^2\right) & \sigma_1\left(\alpha\theta^3\right) \\ \sigma_2(\alpha) & \sigma_2(\alpha\theta) & \sigma_2\left(\alpha\theta^2\right) & \sigma_2\left(\alpha\theta^3\right) \\ \sigma_3(\alpha) & \sigma_3(\alpha\theta) & \sigma_3\left(\alpha\theta^2\right) & \sigma_3\left(\alpha\theta^3\right) \\ \sigma_4(\alpha) & \sigma_4(\alpha\theta) & \sigma_4\left(\alpha\theta^2\right) & \sigma_4\left(\alpha\theta^3\right) \end{bmatrix}$$

We can verify that, after lattice basis reduction [11] using the following unimodular matrix,

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 \\ -1 & -3 & 1 & 1 \end{pmatrix}$$

the generator matrix given in eq. (2) is unitary.

We will consider now a cyclic division algebra $\mathcal{A}$ over $\mathbb{K}$. let $\left\{1, e, e^2, e^3\right\}$ be a base of $\mathcal{A}$, such that $\gamma \in \mathbb{C}$ and $\gamma, \gamma^2, \gamma^3$ are not algebraic norm of any elements of $\mathbb{K}$ [12, 8],

$$e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma & 0 & 0 & 0 \end{bmatrix}$$

[1] The fields $\mathbb{K}$ and $\mathbb{L}$ coincide abstractly, it is only for convenience of exposition that we use distinct notations

$$R_3' = \begin{pmatrix} 1.03826 + 0.32732i & -0.462069 - 0.145674i & 0.832620 + 0.262495i \\ -0.11412 + 0.32732i & -0.142307 + 0.408169i & 0.063332 - 0.181652i \\ 0.39873 + 0.32732i & -0.718498 - 0.589822i & -0.895953 - 0.735496i \end{pmatrix} \quad (5)$$

We can represent all elements in $\mathcal{A}$ by 4×4 matrices, $\mathbf{X} = \sum_{i=1\ldots4} D_i \cdot e^{i-1}$, where $D_i, i = 1\ldots4$ are diagonal matrices with elements in $\mathbb{K}$. We define our infinite code $\mathcal{C}_\infty$ [1] as a subalgebra of $\mathcal{A}$, obtained by restricting elements of $D_i, i = 1\ldots4$ to $\mathcal{I}_\mathbb{K}$. Let $\mathbf{s}_i, i = 1\ldots4$ be a four-element vector in $\mathbb{Z}[i]$. Codewords of $\mathcal{C}_\infty$ are given by

$$\mathbf{X} = \sum_{i=1\ldots4} \text{diag}\left(R_4' \cdot \mathbf{s}_i\right) \cdot e^{i-1}$$

We now have to choose γ such that none of $\gamma, \gamma^2, \gamma^3$ are algebraic norms of any element of $\mathbb{K}$. Also, the norm of γ be equal to 1 in order to guarantee the same average transmitted energy from each antenna, at each channel use. This limits our choice to $\gamma = \pm1, \pm i$. In fact, we can prove, by using Class Field Theory tools that $\gamma = i$, is **the** solution. This choice of γ ensures that the determinant of any codeword $\mathbf{X}$ takes values in the discrete set $\frac{N_{\mathbb{K}/\mathbb{Q}(i)}(\alpha)}{15^2} \cdot \mathbb{Z}[i]$. In fact $\mathcal{A} = (\mathbb{K}/\mathbb{Q}(i), \sigma, i)$ is a cyclic division algebras, as defined in [12, 8], and for $X \in \mathcal{A}, N_{\mathbb{K}/\mathbb{Q}(i)}(X) = det(X) \in \mathbb{Q}(i)$. Then if we take X such that all his entries are from $\mathcal{O}_\mathbb{K}$, $det(X) \in \mathcal{O}_\mathbb{F} = \mathbb{Z}(i)$.

To calculate the minimum determinant we will rewrite R

$$R = \text{diag}\left(\sigma_i(\alpha)\right) \cdot R_{\mathcal{O}_\mathbb{K}}$$

with $R_{\mathcal{O}_\mathbb{K}}$ the generator matrix of $\Lambda\left(\mathcal{O}_\mathbb{K}\right)$, obtained by applying the canonical embedding σ to $\mathcal{B}_\mathbb{K}$. Let

$$P = \sum_{i=1\ldots4} \text{diag}(R_{\mathcal{O}_\mathbb{K}} \cdot \mathbf{s}_i) \cdot e^{i-1}$$

Then minimum determinant is equal to

$$\det(\mathbf{X}) = \tfrac{1}{15^2} \cdot \det(\text{diag}(\sigma_i(\alpha))) \cdot \det(P)$$

As $\det(P)$ is a sum of relative norms and traces in $\mathcal{O}_\mathbb{K}$, it takes values in $\mathbb{Z}[i]$ and its minimum modulus is equal to 1. We conclude that

$$\delta_{min}(\mathcal{C}_\infty) = \tfrac{1}{15^4} \cdot \left|N_{\mathbb{K}/\mathbb{Q}(i)}(\alpha)\right|^2 = \tfrac{1}{1125}$$

3. 3×3 STBC construction

The construction of 3×3 STBC is similar to the 4×4 STBC, with the difference that we consider the field extension $\mathbb{Q}(j)$, $j = \exp(i\frac{2\pi}{3})$, instead of $\mathbb{Q}(i)$. We construct a rotated version of $\mathbb{Z}[j]^3 = A_2^3$ where A_2 is the hexagonal lattice.

Let $\mathbb{K} = \mathbb{Q}(j, 2\cos(\frac{2\pi}{7}))$ be an extension of $\mathbb{Q}(j)$ of degree 3. We denote $\mathbb{L}$ the absolute extension on $\mathbb{Q}$.

The relative discriminant of $\mathbb{K}$ is $d_\mathbb{K} = 49 = 7^2$, while its absolute discriminant is $d_\mathbb{L} = -1 \cdot 3^3 \cdot 7^4$.

An ideal $\mathcal{I}_\mathbb{K}$ of norm 7 can be found from the following ideal factorization

$$(7)\mathcal{O}_\mathbb{K} = \mathcal{I}_7 \cdot \overline{\mathcal{I}_7}$$

Let's consider $\mathcal{I}_\mathbb{K} = \mathcal{I}_7$. It is a principal ideal, so $\mathcal{I}_\mathbb{K} = (\alpha)$ with $\alpha = 1 + j + \theta$.

The relative basis of $\mathcal{I}_\mathbb{K}$ is $\mathcal{B}_{\mathcal{I}_\mathbb{K}} = \left\{\alpha, \alpha\theta, \alpha\theta^2\right\}$. By applying the canonical embedding σ to $\mathcal{B}_{\mathcal{I}_\mathbb{K}}$we obtain the generator matrix of $\Lambda\left(\mathcal{I}_\mathbb{K}\right)$, which we normalize by $\frac{1}{\sqrt{7}}$.

$$R = \tfrac{1}{\sqrt{7}} \cdot \begin{bmatrix} \sigma_1(\alpha) & \sigma_1(\alpha\theta) & \sigma_1\left(\alpha\theta^2\right) \\ \sigma_2(\alpha) & \sigma_2(\alpha\theta) & \sigma_2\left(\alpha\theta^2\right) \\ \sigma_3(\alpha) & \sigma_3(\alpha\theta) & \sigma_3\left(\alpha\theta^2\right) \end{bmatrix}$$

We can verify that, after lattice basis reduction [11] using the following unimodular matrix,

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

the generator matrix of (5) gives a rotated version of A_2^3. Using the cyclic division algebras $\mathcal{A} = (\mathbb{K}/\mathbb{Q}(j), \sigma, j)$, we define the 3×3 STBC in a similar manner as for the case $M = 4$. We can show that

$$\delta_{min}(\mathcal{C}_\infty) = \tfrac{1}{7^3} \cdot \left|N_{\mathbb{K}/\mathbb{Q}(j)}(\alpha)\right|^2 = \tfrac{1}{49}$$

4. 6×6 STBC construction

As the construction is the same as in the 3×3 STBC with $\mathbb{K} = \mathbb{Q}(j, 2\cos(\frac{\pi}{14}))$, we will just give the field extension and the corresponding ideal, necessary to obtain a rotated version of $\mathbb{Z}[j]^6$. The absolute discriminant of $\mathbb{K}$ is $d_\mathbb{L} = 2^{12} \cdot 3^6 \cdot 7^{10}$. An ideal $\mathcal{I}_\mathbb{K}$ with absolute norm 7 can be found from the following ideal factorization

$$(7)\mathcal{O}_\mathbb{K} = \mathcal{I}_7 \cdot \overline{\mathcal{I}_7}$$

We consider the ideal $\mathcal{I}_\mathbb{K} = \mathcal{I}_7$

Then, the 6×6 STBC is a subalgebra of the cyclic division algebra $\mathcal{A} = (\mathbb{K}/\mathbb{Q}(j), \sigma, -j)$, and we prove that

$$\delta_{min}(\mathcal{C}_\infty) = \tfrac{1}{14^6} \cdot \left|N_{\mathbb{K}/\mathbb{Q}(j)}(\alpha)\right|^2 = \tfrac{1}{2^6 7^5}$$

5. Simulation results

We have simulated the complete MIMO transmission scheme using the constructed Space-Time codes, represented by equation (1). The transmitted symbols belong to q-QAM (4 antennas) or q-HEX (3 and 6 antennas) constellations, with $q = 4, 8, 16$, and average energy per bit fixed to 1, q-HEX constellation will be a finite subset of A_2. For the decoding, we use the modified version of the Sphere-Decoder presented in [13].

In Fig. 1 and 2, we show the codeword error rates for our new 4×4 and 3×3 STBC (NC), and the best previously known 4×4 and 3×3 STBC (BPC) in [6, 7], as a function of the E_b/N_0. We can observe that both codes have almost the same performances. We remark in Fig. 1 that the NC is better than BPC for $8, 16$-HEX constellations. For the 4×4 STBC, the BPC have better performances than the NC but the gap decreases from 4-QAM to 16-QAM.

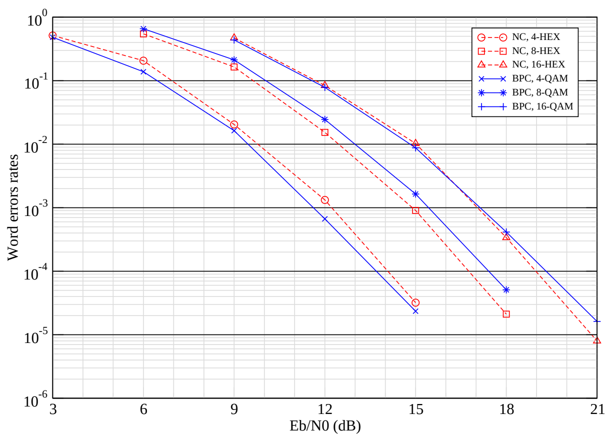

Figure 1: New codes (NC) vs. best previously known codes (BPC) for 3 transmit antennas

6. Conclusion

We present, in this paper, new algebraic constructions of full-rate, fully diverse 3×3, 4×4 and 6×6 Space-Time Codes, having a constant minimum determinant as the spectral efficiency increases.

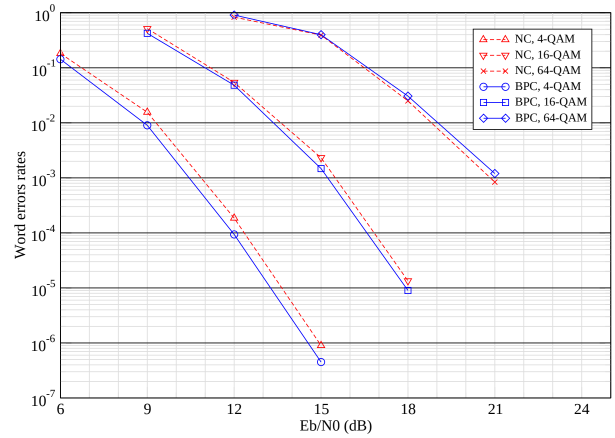

Figure 2: New codes (NC) vs. best previously known codes (BPC) for 4 transmit antennas

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