Applications of the Golden Code

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***Abstract*— The Golden code is a full rate, full diversity 2×2 linear dispersion space-time block code (STBC) that was constructed using cyclic division algebras. The underlaying algebraic structure provides the exceptional properties of the Golden code: cubic shaping and non-vanishing minimum determinant.**

In this paper, we first give a basic introduction about the Golden code. We discuss how to use the Golden code in a practical concatenated coding scheme for 2×2 MIMO systems based on OFDM, such as the ones proposed for high rate indoor wireless LAN communications (e.g. 802.11n).

The proposed bandwidth efficient concatenated scheme uses the Golden code as an inner multidimensional modulation and a trellis code as outer code. Lattice set partitioning is designed in order to increase the minimum determinant. A general framework for code construction and optimization is developed. It is shown that this Golden Space-Time Trellis Coded Modulation scheme can provide excellent performance for high rate applications.

I. Introduction

The Golden code was proposed in [1] as an optimum 2×2 linear dispersion space-time block codes for a 2×2 MIMO system. In this paper, we analyze the application of the Golden code to wireless networks for multimedia traffic, which demand very high spectral efficiency coding schemes with low packet delay. In order to achieve the performance, the combination of multiple transmit/receive antennas with OFDM has attracted most interest for next generation physical layers. Currently, channel bandwidths allocated in the most recent standards are about 20MHz. In order to achieve bit rates of several hundreds Mbit/s, a few multiple antennas are not sufficient, so that bandwidth efficient coding becomes necessary.

Wireless channels are commonly modeled as *block fading channels*. Let W denote the total channel bandwidth and T_ℓ denote the maximum latency that can be tolerated by the real time applications. Let T_c be the channel *coherence time* and B_c the channel *coherence bandwidth*. In the block fading model the channel coefficients are assumed to be constant over a frame of duration T_c and vary independently from one frame to another. Similarly, for the frequency domain channel transfer function is assumed to be constant over a subband of width B_c and vary independently from one subband to another.

In reality, indoor wireless channels are mostly impaired by multipath, which results in a relatively small B_c. On the other hand the reduced mobility within the indoor environment results in a relatively large T_c. Using the OFDM technique we discretize the time-frequency plane (T_ℓ, W) into time-frequency slots of size $(\Delta t, \Delta f)$. Then $N_t = T_\ell/\Delta t$ denotes the number of OFDM symbols than can be transmitted and $N_f = W/\Delta f$ denotes the number of subcarriers within each OFDM symbol.

With this scenario it is common practice to design systems where $T_\ell \leq T_c$ and $\Delta f \approx B_c$, which results in a *slow* fading in time and a *fast* fading in frequency (see Fig. 1). In this case each frame will see a non time-varying transfer function $H(f,t) = H(f)$. Depending on the application, a coded system will employ a certain number N_s of time-frequency slots within a frame to transmit one codeword. We will assume that N_s divides exactly the total number $N_t N_f$ of time-frequency slots within a frame, i.e., $N_t N_f = K N_s$, where K is the number of codewords per frame.

In the 2×2 MIMO case, when the antenna separation is sufficiently large, we have $n_t n_r = 4$ independent channels that can be exploited to gain diversity. In order to transmit a 2×2 Golden codeword X we need $N_s = 2$ time-frequency slots. Since the Golden code is a space-time code designed for a slow fading channel we must choose two slots in consecutive OFDM symbols that have a non time-varying channel. The 2×2 received signal matrix can be written as

$$Y = HX + Z$$

where Z is the additive white Gaussian noise matrix and H is the 2×2 matrix which is assumed to be constant for the two channel uses.

Assume we want to transmit codewords $\mathbf{X} = [X_1, \ldots, X_L] \in \mathbb{C}^{2\times 2L}$ which are obtained by concatenating the Golden code with some outer code. Given that the first row of $\mathbf{X}$ contains the time-frequency samples of the signal $X_1(f,t)$ sent over the first antenna and the second row the ones of the signal $X_2(f,t)$ sent over the second antenna, we have different options for positioning the components of $\mathbf{X}$ in the time-frequency frame.

Figure 1 shows the case where $K = N_f$ codewords are sent over $2L$ consecutive time slots within the same OFDM frequency subband. If $2L\Delta t \approx T_c$ we have the slow fading channel given by

$$\mathbf{Y} = \mathbf{HX} + \mathbf{Z} \tag{1}$$

where $\mathbf{Z} \in \mathbb{C}^{2\times 2L}$ is the complex white Gaussian noise with i.i.d. samples $\sim \mathcal{N}_{\mathbb{C}}(0, N_0)$, $\mathbf{H} \in \mathbb{C}^{2\times 2}$ is the channel matrix,

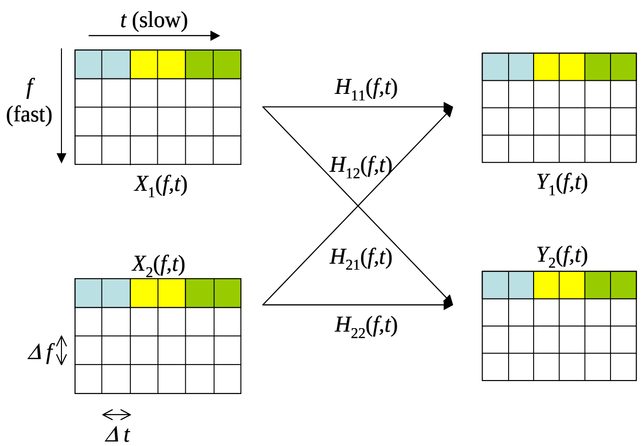

Fig. 1. Space-Time-Frequency codeword allocation in a 2×2 MIMO system: the codeword is transmitted through a slow fading channel.

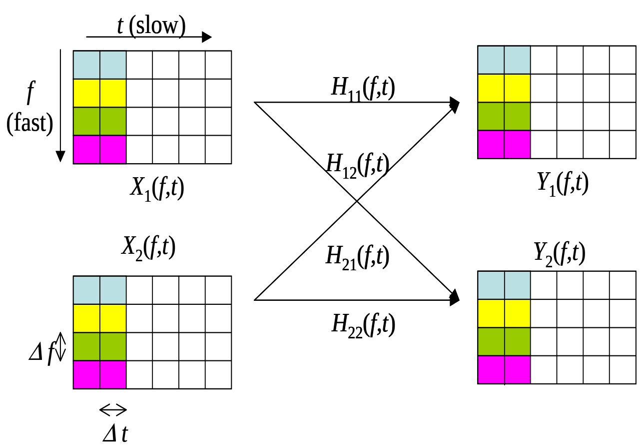

Fig. 2. Space-Time-Frequency codeword allocation in a 2×2 MIMO system: the codeword is transmitted through a fast fading channel.

which is constant during a frame and varies independently from one frame to another. The elements of $\mathbf{H}$ are assumed to be i.i.d. circularly symmetric Gaussian random variables $\sim \mathcal{N}_{\mathbb{C}}(0,1)$.

Figure 2 shows the case where each codeword is sent within two consecutive OFDM symbols. A total of $K = N_t/2$ codewords are sent over L frequency slots. If $\Delta f \approx B_c$ we have the fast fading channel given by

$$Y_k = H_k X_k + Z_k \qquad k = 1, \ldots, L \tag{2}$$

In practice, a frequency interleaver is often inserted in order to provide better independence between the channel coefficient matrices H_k in different subbands. For convenience we will assume that $L = N_f$ but other codeword lengths can be easily adapted to the frame if L is an integer fraction or multiple of N_f.

The channel is assumed to be known at the receiver. This can be obtained by sending some pilot symbols to estimate the channel at the receiver. Note that at least one OFDM symbol per coherence time is needed in order to track the channel variations.

The careful concatenation of the Golden code with an outer trellis code provides a robust solution for high rate transmission over such channel. We will discuss different solutions for the application of Golden space-time trellis coded modulation (GST-TCM).

As an inner code, the Golden code guarantees full diversity for any spectral efficiency, and the outer trellis code is used to improve the coding gain. We note how the NVD property for the inner code is essential when using a TCM scheme: such schemes usually require a constellation expansion, which will not suffer from a reduction of the minimum determinant.

We therefore develop a systematic design approach for GST-TCM schemes. Lattice set partitioning, combined with a trellis code, is used to increase the minimum determinant. The Viterbi algorithm is used for trellis decoding, where the branch metrics are computed by using a lattice sphere decoder [6] for the inner code. It is shown that the proposed TCMs achieve significant performance gains.

The rest of the paper is organized as follows. Section 2 introduces the system model. Section 4 reviews the Golden code. Section 3 presents code design criteria. Section 4 presents the GST-TCM scheme. Conclusions are drawn in Section 5.

The following notations are used in the paper. Let T denote transpose and $\dagger$ denote Hermitian transpose. Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ and $\mathbb{Z}[i]$ denote the ring of rational integers, the field of rational numbers, the field of complex numbers, and the ring of Gaussian integers, where $i^2 = -1$. Let $GF(2) = \{0,1\}$ denote the binary Galois field. Let L/K denote a field extension. Let $\mathcal{A}$ denote the cyclic algebra. Let $\mathbb{Q}(\theta)$ denote an algebraic number field generated by the primitive element θ. The real and imaginary parts of a complex number are denoted by $\Re(\cdot)$ and $\Im(\cdot)$. The $m \times m$ dimensional identity matrix is denoted by $\mathbf{I}_m$. The $m \times n$ dimensional zero matrix is denoted by $\mathbf{0}_{m \times n}$. The Frobenius norm of a matrix is denoted by $\|\cdot\|_F$. Let $\mathbb{Z}^8$ be the 8-dimensional integer lattice and E_8 be the Gosset lattice, the densest sphere packing in 8 dimensions [13].

In this paper, we use Q-QAM constellations with $Q = 2^\eta$. We assume the constellation is scaled to match $\mathbb{Z}[i] + (1+i)/2$, i.e., the minimum Euclidean distance is set to 1 and it is centered at the origin. For example, the average energy is $E_s = 0.5, 1.5, 2.5, 5, 10.5$ for $Q = 4, 8, 16, 32, 64$.

Signal to noise ratio is defined as $\text{SNR} = n_t E_b / N_0$, where $E_b = E_s/q$ is the energy per bit and q denotes the number of information bits per symbol. We have $N_0 = 2\sigma^2$, where σ^2 is the noise variance per real dimension, which can be adjusted as $\sigma^2 = (n_t E_b/2) 10^{(\text{-SNR}/10)}$.

II. The Golden Code

The Golden code was proposed in [1] as an optimum 2×2 linear dispersion space-time block codes for a 2×2 MIMO system. The code is constructed using a particular family of *division algebras*, named *cyclic division algebras*. It is built over a quadratic extension of the base field $\mathbb{Q}(i)$, where $i^2 = -1$, thereby enabling to use arbitrary Q-QAM constellations. The inherent integer lattice structure provides efficient constellation shaping, leading to the *information lossless property* [5]. The base field $\mathbb{Q}(i)$ also provides the *non-vanishing determinant* (NVD) property for the Golden code, i.e., the minimum determinant remains constant for any QAM size. It was shown in [3] that the NVD guarantees to achieve the fundamental performance limit of the multiple-input multiple-output (MIMO) systems, given by the diversity-multiplexing tradeoff (DMT) [4].

Following [1] and [2], we review the algebraic properties of the Golden code, which is built using the cyclic algebra

$$\mathcal{A} = (L/K = \mathbb{Q}(i, \sqrt{5})/\mathbb{Q}(i), \sigma, i),$$

with $\sigma : \sqrt{5} \mapsto -\sqrt{5}$. We have that the ring of integers of the field L is

$$\mathcal{O}_L = \{a + b\theta \mid a, b \in \mathbb{Z}[i]\},$$

where $\theta = \frac{1+\sqrt{5}}{2}$. Before shaping, a codeword from this algebra is of the form

$$\left[\begin{array}{cc} a + b\theta & c + d\theta \\ i(c + d\sigma(\theta)) & a + b\sigma(\theta) \end{array} \right],$$

with four information symbols $a, b, c, d \in \mathbb{Z}[i]$.

Since i is not a norm of any element of L, $\mathcal{A}$ is a cyclic division algebra [2]. By definition, the codebook obtained is linear, full rate (since it contains four information symbols a, b, c, d), and fully diverse.

A. The cubic shaping

Let us see now how to add the cubic shaping on the codebook built on $\mathcal{A}$, which provides the information lossless property. We look at the $\mathbb{Z}[i]$-lattice that is generated on each layer of the codeword by the matrix

$$\left(\begin{array}{cc} 1 & \theta \\ 1 & \sigma(\theta) \end{array} \right)$$

This matrix is not unitary, hence we have an energy shaping loss. We now show how to fix this problem without loosing all the other algebraic properties. This can be obtained by a lattice Λ with a generator matrix of the form

$$M = \left(\begin{array}{cc} \alpha & 0 \\ 0 & \sigma(\alpha) \end{array} \right) \left(\begin{array}{cc} 1 & \theta \\ 1 & \sigma(\theta) \end{array} \right) = \left(\begin{array}{cc} \alpha & \alpha\theta \\ \sigma(\alpha) & \sigma(\alpha)\sigma(\theta) \end{array} \right).$$

where $\alpha \in \mathcal{O}_L$ has to be selected in order to make M unitary. The determinant of the lattice Λ can be written as

$$\begin{array}{rcl} \det(\Lambda) & = & |N_{L/K}(\alpha)|^2 |d_{\mathbb{Q}(\sqrt{5})}| \\ & = & 5|N_{L/K}(\alpha)|^2. \end{array}$$

A necessary condition to have the lattice $\mathbb{Z}[i]^2$ is that $\det(\Lambda)$ is a square integer. We thus look for an element α such that $|N_{L/K}(\alpha)|^2 = 5$. In order to find such element, we look at the factorization of 5 in $\mathcal{O}_L$:

$$5 = (1 + i - i\theta)^2 (1 - i + i\theta)^2.$$

We thus choose $\alpha = 1 + i - i\theta$. Let us now check we indeed get the right lattice. Using its generator matrix M a direct computation shows that $M^\dagger = 5I_2$. Thus $\frac{1}{\sqrt{5}}M$ is a unitary matrix, yielding the shaping property within the first layer. The other layer also has the cubic shaping so that the entire codeword has the

A codeword $\mathbf{X}$ belonging to the Golden code has thus, adding the shaping property, the form

$$\mathbf{X} = \frac{1}{\sqrt{5}} \left[\begin{array}{cc} \alpha(a + b\theta) & \alpha(c + d\theta) \\ i\sigma(\alpha)(c + d\sigma(\theta)) & \sigma(\alpha)(a + b\sigma(\theta)) \end{array} \right]$$

where a, b, c, d are QAM symbols.

Recall that when a, b, c, d can take any value in $\mathbb{Z}[i]$, we say that we have an *infinite code* $\mathcal{C}_\infty$. This terminology recalls the case where finite signal constellations are carved from infinite lattices.

B. The Minimum Determinant

Let us now compute the minimum determinant of the infinite code. Since

$$\mathbf{X} = \frac{1}{\sqrt{5}} \left[\begin{array}{cc} \alpha & 0 \\ 0 & \sigma(\alpha) \end{array} \right] \left[\begin{array}{cc} a + b\theta & c + d\theta \\ i(c + d\sigma(\theta)) & a + b\sigma(\theta) \end{array} \right],$$

and since $\alpha\sigma(\alpha) = 2 + i$, we have

$$\begin{aligned} \det(\mathbf{X}) &= \frac{2+i}{5} [(a + b\theta)(a + b\sigma(\theta)) - i(c + d\theta)(c + d\sigma(\theta))] \\ &= \frac{1}{2-i} [(a^2 + ab - b^2 - i(c^2 + cd - d^2)]. \end{aligned}$$

By definition of a, b, c, d, we have that the non trivial minimum of $|a^2 + ab - b^2 - i(c^2 + cd - d^2)|^2$ is 1, thus

$$\delta_{\min}(\mathcal{C}_\infty) = \min_{\mathbf{X} \neq \mathbf{0}} |\det(\mathbf{X})|^2 = \frac{1}{5}.$$

Thus the minimum determinant of the infinite code is bounded away from zero, as required by the NVD property.

Finally, we note that in the second row of the codeword $\mathbf{X}$ the factor i guarantees a uniform average transmitted energy from both antennas in both channel uses, since $|i|^2 = 1$.

III. Code Design Criteria

Let us first consider the slow fading channel in (1). Assuming that a codeword $\mathbf{X}$ is transmitted, the maximum-likelihood receiver might decide erroneously in favor of another codeword $\hat{\mathbf{X}}$. Let r denote the rank of the *codeword difference matrix* $\mathbf{X} - \hat{\mathbf{X}}$. Since the Golden code is a full rank code, we have $r = n_t = 2$.

Let $\lambda_j, j = 1, \ldots, r$, be the eigenvalues of the *codeword distance matrix* $\mathbf{A} = (\mathbf{X} - \hat{\mathbf{X}})(\mathbf{X} - \hat{\mathbf{X}})^\dagger$. Let $\Delta = \prod_{j=1}^{n_t} \lambda_j$ be the determinant of the codeword distance matrix $\mathbf{A}$ and $\Delta_{\min}$ be the corresponding *minimum determinant*, which is defined as

$$\Delta_{\min}^{(s)} = \min_{\mathbf{X} \neq \hat{\mathbf{X}}} \det(\mathbf{A}). \qquad (3)$$

The pairwise error probability (PWEP) is upper bounded by

$$P\left(\mathbf{X} \to \hat{\mathbf{X}}\right) \leq \left(\Delta_{\min}^{(s)}\right)^{-n_r} \left(\frac{E_s}{N_0}\right)^{-n_t n_r} \qquad (4)$$

where $n_t n_r$ is the *diversity gain* and $(\Delta_{\min})^{1/n_t}$ is the *coding gain* [7]. In the case of linear codes analyzed in this paper, we can simply consider the all-zero codeword matrix and we have

$$\Delta_{\min}^{(s)} = \min_{\mathbf{X}\neq\mathbf{0}_{2\times 2L}} \left|\det\left(\mathbf{X}\mathbf{X}^{\dagger}\right)\right|^{2}. \tag{5}$$

Let us now consider the fast fading channel in (2). In this case the PWEP is upper bounded by

$$P\left(\mathbf{X}\rightarrow\hat{\mathbf{X}}\right) \leq \left(\Delta_{\min}^{(f)}\right)^{-n_r L'} \left(\frac{E_s}{N_0}\right)^{-n_t n_r L'} \tag{6}$$

where L' is minimum number of non-zero determinants

$$\det\left((X_k-\hat{X}_k)(X_k-\hat{X}_k)^{\dagger}\right) \tag{7}$$

and the minimum determinant in the case of linear codes is given by

$$\Delta_{\min}^{(f)} = \min_{\mathbf{X}\neq\mathbf{0}_{2\times 2L}} \prod_{\det(X_k X_k^{\dagger})\neq 0}^{(L')} \det\left(X_k X_k^{\dagger}\right) \tag{8}$$

Note how L' mimics the role of the Hamming distance which was used in the design of STTC for fast fading proposed by Tarokh *et al.* in [7]. The diversity order of the code over the fast fading channel is thus increased by a factor L'.

In order to compare two coding schemes for the same type of channel in a 2×2 MIMO system, supporting the same information bit rate, but different minimum determinants ($\sqrt{\Delta_{\min,1}}$ and $\sqrt{\Delta_{\min,2}}$) and different constellation energies ($E_{s,1}$ and $E_{s,2}$), we define the asymptotic coding gain as

$$\gamma_{as} = \frac{\sqrt{\Delta_{\min,1}}/E_{s,1}}{\sqrt{\Delta_{\min,2}}/E_{s,2}} \tag{9}$$

Performance of an uncoded Golden code scheme, where the all X_k in $\mathbf{X}$ are independently selected from the Golden code, can be simply analyzed for $L=1$ on both slow and fast fading. Since the Golden code $\mathcal{G}$ has minimum determinant is $\delta_{\min}=\frac{1}{5}$ we have $\Delta_{\min}^{(s)}=\Delta_{\min}^{(f)}=\delta_{\min}$.

In general, we consider $L>1$ and the minimum determinant for slow fading can be written as

$$\Delta_{\min}^{(s)} = \min_{\mathbf{X}\neq\mathbf{0}_{2\times 2L}} \det(\mathbf{X}\mathbf{X}^{\dagger}) = \min_{\mathbf{X}\neq\mathbf{0}_{2\times 2L}} \det\left(\sum_{k=1}^{L}\left(X_k X_k^{\dagger}\right)\right). \tag{10}$$

A code design criterion attempting to maximize $\Delta_{\min}^{(s)}$ is hard to exploit, due to the non-additive nature of the determinant metric in (10). Since $X_t X_t^{\dagger}$ are positive definite matrices, we use the following determinant inequality [14]:

$$\Delta_{\min}^{(s)} \geq \min_{\mathbf{X}\neq\mathbf{0}_{2\times 2L}} \sum_{k=1}^{L} \det\left(X_k X_k^{\dagger}\right) = \Delta_{\min}^{\prime(s)}. \tag{11}$$

Note that only L' terms will be non zero in the above sum. Similarly, we have L' terms in the product giving the $\Delta_{\min}^{(f)}$ for the fast fading case.

In order to optimize the performance of the coding scheme we focus on the lower bound $\Delta_{\min}^{\prime(s)}$ for slow fading and $\Delta_{\min}^{(f)}$ for fast fading. In particular we will design trellis codes that attempt to maximize these two quantities, by using set partitioning to increase the number L' and the magnitude of the non zero terms $\det\left(X_k X_k^{\dagger}\right)$. This will yield codes that are robust to both types of channels.

IV. Golden Space-Time Trellis Coded Modulation

In this section, we propose a systematic design approach for Golden Space-Time Trellis coded (GST-TCM). We analyze the design problem of this scheme by using Ungerboeck style set partitioning rules for coset codes [9–11]. The design criterion for the trellis code is developed in order to maximize the quantity $\Delta'_{\min}$ denoting either $\Delta_{\min}^{\prime(s)}$ or $\Delta_{\min}^{(f)}$. This results in the maximum lower bound on the asymptotic coding gain of the GST-TCM over the uncoded Golden code scheme

$$\gamma_{as} \geq \frac{\sqrt{\Delta'_{\min}}/E_{s,2}}{\sqrt{\delta_{\min}}/E_{s,1}}. \tag{12}$$

Before we design the coding scheme, we briefly recall the set partition chain of the Golden code given in [12].

The Golden subcodes – Let us consider a subcode $\mathcal{G}_k \subseteq \mathcal{G}$ for $k=1,\ldots,4$, obtained by

$$\mathcal{G}_k = \{XB^k, X\in\mathcal{G}\}, \tag{13}$$

where

$$B = \begin{bmatrix} i(1-\theta) & 1-\theta \\ i\theta & i\theta \end{bmatrix}. \tag{14}$$

This provides the minimum square determinant $2^k\delta_{\min}$ (see Table I). It can be shown that the codewords of $\mathcal{G}_k$, when vectorized, correspond to different sublattices of $\mathbb{Z}^8$ that form the lattice partition chain

$$\mathbb{Z}^8 \supset D_4^2 \supset E_8 \supset L_8 \supset 2\mathbb{Z}^8 \tag{15}$$

where D_4^2 is the direct sum of two four-dimensional Shäfli lattices, E_8 is the Gosset lattice and L_8 is a lattice of index 64 in $\mathbb{Z}^8$. Any two consecutive lattices $\Lambda_k \supset \Lambda_{k+1}$ in this chain form a four way partition, i.e., the quotient group Λ_k/Λ_{k+1} has order 4. Let $[\Lambda_k/\Lambda_{k+1}]$ denote the set of coset leaders of the quotient group Λ_k/Λ_{k+1}.

The lattices in the partition chain can be obtained by Construction A [13], using the nested sequence of linear binary codes C_k listed in Table I. Let G_k denote the generator matrix of the code C_k for $k=1,2,3$. We have

$$G_1 = \begin{bmatrix} 1&0&0&1&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&1&1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0&1&0&1&0&1&0&1\\ 0&0&1&1&0&0&1&1\\ 0&0&0&0&1&1&1&1\\ 1&1&1&1&1&1&1&1 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 0&0&0&0&1&1&1&1\\ 1&1&1&1&1&1&1&1 \end{bmatrix}$$

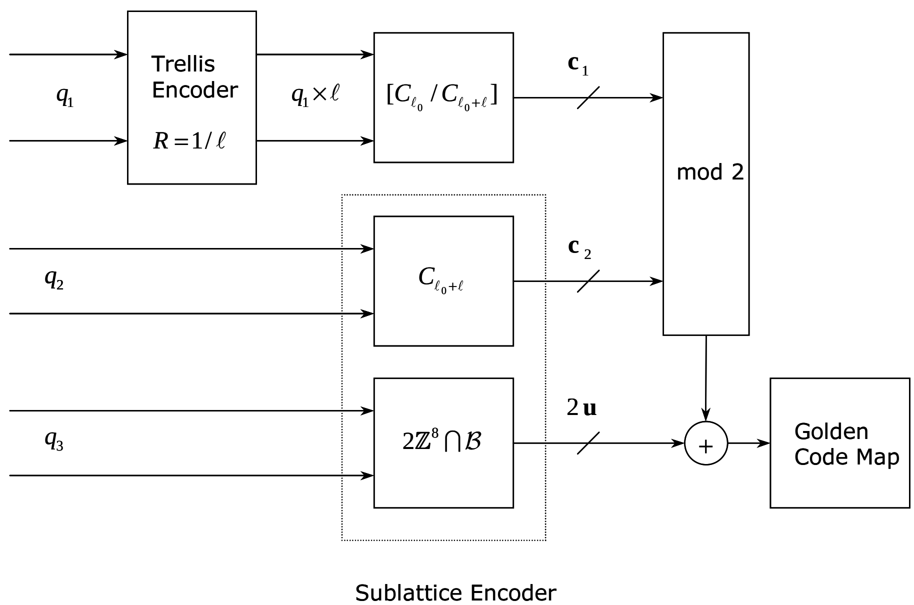

Fig. 3. General encoder structure of the concatenated scheme.

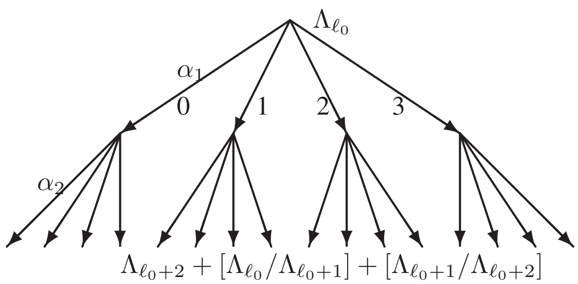

Fig. 4. Two level ($\ell = 2$) partition tree of Λ_{ℓ_0} into 16 cosets of Λ_{ℓ_0+2}.

Level	Subcode	Lattice	Binary code	$\Delta_{\min}$
0	$\mathcal{G}$	$\mathbb{Z}^8$	$C_0 = (8, 8, 1)$	$\delta_{\min}$
1	$\mathcal{G}_1$	D_4^2	$C_1 = (8, 6, 2)$	$2\delta_{\min}$
2	$\mathcal{G}_2$	E_8	$C_2 = (8, 4, 4)$	$4\delta_{\min}$
3	$\mathcal{G}_3$	L_8	$C_3 = (8, 2, 4)$	$8\delta_{\min}$
4	$\mathcal{G}_4 = 2\mathcal{G}$	$2\mathbb{Z}^8$	$C_4 = (8, 0, \infty)$	$16\delta_{\min}$

TABLE I

THE GOLDEN CODE PARTITION CHAIN WITH CORRESPONDING LATTICES, BINARY CODES, AND MINIMUM SQUARED DETERMINANTS.

Following the track of [9–11], we consider a partition tree of the Golden code of depth ℓ (see Fig. 4). From a nested subcode sequence $\mathcal{G} \supseteq \mathcal{G}_{\ell_0} \supset \mathcal{G}_{\ell_0+1} \supset \cdots \supset \mathcal{G}_{\ell_0+\ell}$, we have the corresponding lattice partition chain $\mathbb{Z}^8 \supseteq \Lambda_{\ell_0} \supset \Lambda_{\ell_0+1} \supset \cdots \supset \Lambda_{\ell_0+\ell}$ where

$$\begin{aligned}\Lambda_{\ell_0} &= \Lambda_{\ell_0+1} + [\Lambda_{\ell_0}/\Lambda_{\ell_0+1}] = \cdots \\ &= \Lambda_{\ell_0+\ell} + [\Lambda_{\ell_0}/\Lambda_{\ell_0+1}] + \cdots + [\Lambda_{\ell_0+\ell-1}/\Lambda_{\ell_0+\ell}] \\ &= \Lambda_{\ell_0+\ell} + [C_{\ell_0}/C_{\ell_0+1}] + \cdots + [C_{\ell_0+\ell-1}/C_{\ell_0+\ell}]\end{aligned}$$

The coset leaders in $[C_k/C_{k+1}]$ form the group of order four $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which is generated by two binary generating vectors $\mathbf{h}_1$ and $\mathbf{h}_2$

$$[C_k/C_{k+1}] = \{b_1\mathbf{h}_1 + b_2\mathbf{h}_2 \mid b_1, b_2 \in GF(2)\}$$

If we consider all the lattices in (15), and the corresponding nested sequence of linear binary codes C_k, we have:

$$\begin{aligned}
[C_0/C_1] &: \begin{cases}\mathbf{h}_1^{(1)} = (0,0,0,0,0,0,0,1)\\ \mathbf{h}_2^{(1)} = (0,0,0,1,0,0,0,0)\end{cases}\\
[C_1/C_2] &: \begin{cases}\mathbf{h}_1^{(2)} = (0,0,0,0,0,0,1,1)\\ \mathbf{h}_2^{(2)} = (0,0,0,0,0,1,0,1)\end{cases}\\
[C_2/C_3] &: \begin{cases}\mathbf{h}_1^{(3)} = (0,1,0,1,0,1,0,1)\\ \mathbf{h}_2^{(3)} = (0,0,1,1,0,0,1,1)\end{cases}\\
[C_3/C_4] &: \begin{cases}\mathbf{h}_1^{(4)} = (0,0,0,0,1,1,1,1)\\ \mathbf{h}_2^{(4)} = (1,1,1,1,1,1,1,1)\end{cases}
\end{aligned} \tag{16}$$

Encoder structure – Fig. 3 shows the encoder structure of the proposed concatenated scheme. The input bits feed two encoders, an upper trellis encoder and a lower lattice encoder.

For two lattices Λ_{ℓ_0} and $\Lambda_{\ell_0+\ell}$, we have the quotient group $\Lambda_{\ell_0}/\Lambda_{\ell_0+\ell}$ with order $N_c = |\Lambda_{\ell_0}/\Lambda_{\ell_0+\ell}| = 4^\ell$, which corresponds to the total number of cosets of the sublattice $\Lambda_{\ell_0+\ell}$ in the lattice Λ_{ℓ_0}. We assume that we have $4q$ input bits. The upper encoder is a trellis encoder that operates on q_1 information bits. Given the relative partition depth ℓ, we select a trellis code rate $R_c = 1/\ell$. The trellis encoder outputs $n_c = q_1/R_c$ bits, which are used by the coset mapper to label a coset leader $\mathbf{c}_1 \in [\Lambda_{\ell_0}/\Lambda_{\ell_0+\ell}]$. The mapping is obtained by the product of the n_c bit vector with a binary coset leader generator matrix H_1 with rows $\mathbf{h}_1^{(\ell_0+1)}, \mathbf{h}_2^{(\ell_0+1)}, \cdots, \mathbf{h}_1^{(\ell_0+\ell)}, \mathbf{h}_2^{(\ell_0+\ell)}$, taken from (16). This implies the choice of $q_1 = 2$.

The lower encoder is a sublattice encoder for $\Lambda_{\ell_0+\ell}$ and operates on q_2+q_3 information bits, where $q_2 = 2\times(4-\ell-\ell_0)$ and $q_3 = 4q - q_1 - q_2$. The q_2 bits label the cosets of $2\mathbb{Z}^8$ in $\Lambda_{\ell_0+\ell}$, i.e., a coset leader $\mathbf{c}_2 \in [\Lambda_{\ell_0+\ell}/2\mathbb{Z}^8]$. The mapping is obtained by the product of the q_2 bit vector with a binary coset leader generator matrix H_2 with rows $\mathbf{h}_1^{(\ell_0+\ell+1)}, \mathbf{h}_2^{(\ell_0+\ell+1)}, \cdots, \mathbf{h}_1^{(4)}, \mathbf{h}_2^{(4)}$, taken from (16). We finally add both coset leaders of $\mathbf{c}_1$ and $\mathbf{c}_2$ modulo 2 to get $\mathbf{c}'$. The q_3 bits go through the $2\mathbb{Z}^8$ encoder and generate vector $2\mathbf{u}$, $\mathbf{u} \in \mathbb{Z}^8$, which is added to $\mathbf{c}'$ (lifted to have integer components) and then mapped to the Golden codeword X_k.

We now focus on the structure of the trellis code to be used. We consider linear convolutional encoders over the quaternary alphabet $\mathbb{Z}_4 = \{0,1,2,3\}$, in order to match the four way partitions. We assume the natural mapping between pairs of bits and quaternary symbols, i.e., $0 \to 00, 1 \to 01, 2 \to 10, 3 \to 11$. Let $\beta \in \mathbb{Z}_4$ denote the input symbol and $\alpha_1, \ldots, \alpha_\ell \in \mathbb{Z}_4$ denote the ℓ output symbols generated by the generator polynomials $g_1(D), \ldots g_\ell(D)$ over $\mathbb{Z}_4$.

Trellis labeling– In order to increase the potential coding gain, the lower bound $\Delta'_{\min}$ should be maximized. Let $\Delta_{\mathrm{par}} = 2^{\ell_0+\ell}\delta_{\min}$ denote the minimum determinant of the trellis parallel transitions corresponding to the Golden codewords in the partition $\Lambda_{\ell_0+\ell} + \mathbf{c}_1$. Let

$$\Delta_{\mathrm{sim}} = \begin{cases} \min\limits_{\mathbf{X} \neq \mathbf{0}_{2\times 2L}} \displaystyle\sum_{k=k_o}^{k_o+L'-1} \det(X_k X_k^\dagger) & \text{(slow)} \\ \min\limits_{\mathbf{X} \neq \mathbf{0}_{2\times 2L}} \displaystyle\prod_{k=k_o}^{k_o+L'-1} \det(X_k X_k^\dagger) & \text{(fast)} \end{cases}$$

denote the minimum determinant on the shortest simple error event, where L' is the length of the shortest simple error event diverging from the zero state at k_o and merging to the zero state at $k_i = k_o + L' - 1$.

The lower bound $\Delta'_{\min}$ in (11) is determined either by the parallel transition error events or by the shortest simple error events in the trellis, i.e.,

$$\Delta'_{\min} = \min\{\Delta_{\mathrm{par}}, \Delta_{\mathrm{sim}}\} \geq$$
$$\geq \min\left\{\Delta_{\mathrm{par}}, \min_{X_{k_o}} \det(X_{k_o} X_{k_o}^\dagger) \begin{array}{c} + \\ \times \end{array} \min_{X_{k_i}} \det(X_{k_i} X_{k_i}^\dagger)\right\}.$$

where the $+$ sign is for slow fading and the $\times$ sign is for fast fading.

Therefore, we focus on maximizing Δ_{sim} in $\Delta'_{\min}$ when selecting the trellis code labeling. We have the following:

Design criterion – *The incoming and outgoing branches for each state should belong to different cosets that have the common father node as deep as possible in the partition tree. This guarantees that simple error events in the trellis give the largest contribution to* $\Delta'_{\min}$.

In order to fully satisfy the above criterion for a given relative partition level ℓ, the minimum number of trellis states should be $N_c = 4^\ell$. In order to reduce complexity we will also consider trellis codes with fewer states. We will see in the following that the performance loss of these suboptimal codes (in terms of the above design rule) is marginal since Δ_{par} is dominating the code performance. Nevertheless, the optimization of Δ_{sim} yields a performance enhancement. In fact, maximizing Δ_{sim} has the effect of minimizing some other relevant term in the determinant spectrum.

Decoding – The decoder is structured as a typical TCM decoder, i.e., a Viterbi algorithm using a branch metric computer. The branch metric computer should output the distance of the received symbol from all the cosets of $\Lambda_{\ell_0+\ell}$ in Λ_{ℓ_0}.

Example – *We use a three level partition* $\mathbb{Z}^8/L_8$ *(*$\ell_0 = 0$ *and* $\ell = 3$*). The 16 and 64 state trellis codes using 16–QAM (*$E_{s,1} = 2.5$*) gain 4.2 and 4.3 dB, respectively, over an uncoded Golden code (*$E_{s,2} = 1.5$*) on the slow fading channel at the rate of 6 bpcu.*

We consider a three level partition with quotient group $\Lambda_{\ell_0}/\Lambda_{\ell_0+\ell} = \mathbb{Z}^8/L_8$ of order $N_c = 64$. The quaternary trellis encoders for 16 and 64 states with rate $R_c = 1/3$ have $q_1 = 2$ input information bits and $n_c = 6$ output bits, which label the coset leaders. The sublattice encoder has $q_2 = 2$ and $q_3 = 8$ input bits, giving a total of $q = (q_1+q_2+q_3)/4 = 12/4 = 3$ information bits per 16–QAM information symbol.

The 16 state GST-TCM has the following generator polynomials: $g_1(D) = D, g_2(D) = D^2, g_3(D) = 1 + D^2$, where D is a delay operator. For the 16 state GST-TCM, at each trellis state, four outgoing branches are labeled with $\alpha_1, \alpha_2, \alpha_3$, corresponding to input $\beta \in \mathbb{Z}_4$. In this case, since α_1 and α_2 are fixed, α_3 varies. This guarantees an increased $\Delta'_{\min}$. The four trellis branches arriving in each state are in cosets of E_8. This does not give the highest possible increase to $\Delta'_{\min}$ since α_2 varies.

We can verify that the shortest simple error event has a length of $L' = 3$ corresponding to the state sequence $0 \to 1 \to 4 \to 0$ and labels $001, 100, 011$. This yields the lower bound of the corresponding asymptotic coding gain

$$\gamma'_{as} = \frac{\sqrt{\min(8\delta_{\min}, 4\delta_{\min} + \delta_{\min} + 2\delta_{\min})/E_{s,1}}}{\sqrt{\delta_{\min}}/E_{s,2}} \to 2.0\,\mathrm{dB}. \tag{17}$$

The above problem suggests the use of a 64 state encoder with the generator polynomials: $g_1(D) = D, g_2(D) = D^2, g_3(D) = 1 + D^3$. In such a case, the output labels $\alpha_1(D_4^2), \alpha_2(E_8)$ are fixed for all outgoing and incoming states. Only $\alpha_3(L_8)$ varies to choose different subgroups from the deepest partition level in this example. This fully satisfies our design rule.

We can see that the shortest simple error event has length $L' = 4$ corresponding to the state sequence $0 \to 1 \to 4 \to 16 \to 0$ and labels $001, 100, 010, 001$. Note that now the output

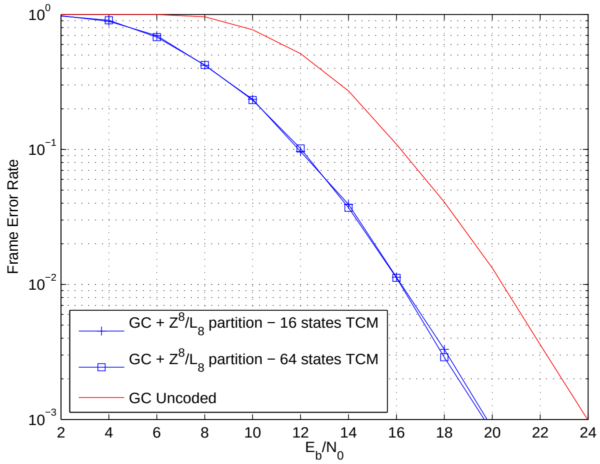

Fig. 5. Performance comparison of a 4-state trellis code using 16-QAM constellation and an uncoded transmission at the rate 6bpcu, $\Lambda_{\ell_0} = \mathbb{Z}^8$, $\Lambda_{\ell_0+\ell} = L_8$, $\ell = 3$, slow fading.

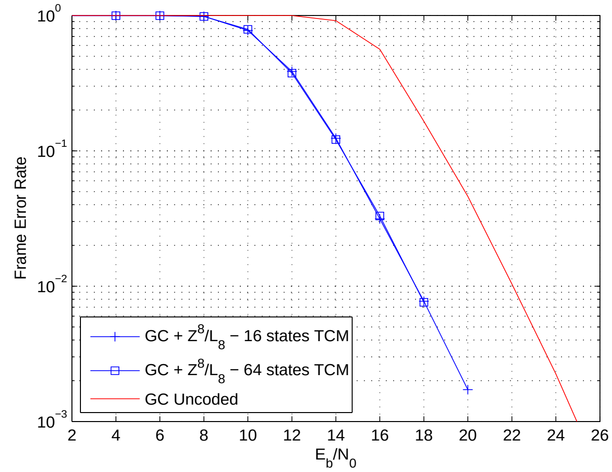

Fig. 6. Performance comparison of 16- and 64-state trellis codes using 16-QAM constellation and an uncoded transmission at the rate 6bpcu, $\Lambda_{\ell_0} = \mathbb{Z}^8$, $\Lambda_{\ell_0+\ell} = L_8$, $\ell = 3$, fast fading.

labels α_1, α_2 are fixed for all outgoing and incoming states. This guarantees both incoming and outgoing trellis branches from each state belong to the cosets that are deepest in the partition tree. This yields the lower bound of the corresponding asymptotic coding gain

$$\gamma'_{as} = \frac{\sqrt{\min(8\delta_{\min}, 4\delta_{\min} + \delta_{\min} + 2\delta_{\min} + 4\delta_{\min})/E_{s,1}}}{\sqrt{\delta_{\min}}/E_{s,2}} \rightarrow 2.3 \text{ dB}.$$

Performance of the proposed codes and the uncoded scheme with 6 bpcu is compared in Fig. 5. We can observe that a 16 state GST-TCM outperforms the uncoded scheme by 4.2 dB and at the FER of 10^{-3}. The 64 state GST-TCM outperforms the uncoded case by 4.3 dB at FER of 10^{-3}.

In the case os fast fading the same codes as above are compared in Fig. 6. It is shown that the 16 and 64 state codes have the same performance and outperform the uncoded Golden code by about 4.5 dB at the FER of 10^{-3}.

By comparing the two figures we see that the performance over the slow fading channel is about 1.5dB better than the one over fast fading. Even if the diversity order in fast fading is larger this will only appear at a much lower error probability. We can conclude that concentrating on the coding gain, independently of the diversity order has a positive effect on the performance in the mid-range SNRs.

V. Conclusions

In this paper, we presented a simple review of the Golden code, showing how the cyclic division algebra guarantees some optimal properties of the Golden code. We then developed a concatenated coding scheme for both slow and fast fading 2×2 MIMO systems based on OFDM. The inner code is the Golden code and the outer code is a trellis code. Lattice set partitioning is designed specifically to increase the minimum determinant of the Golden codewords, which label the branches of the outer trellis code. Viterbi algorithm is applied in trellis decoding, where branch metrics are computed by using a lattice decoder. The general framework for GST-TCM design and optimization is based on Ungerboeck TCM design rules. Finally, it was shown that the design criteria are robust to various channel conditions ranging from slow to fast fading.

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