

Line Codes Generated by Finite Coxeter Groups

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Abstract—Using an algebraic approach based on the theory of Coxeter groups, we design, and describe the performance of, a class of line codes derived from permutation modulation, useful for parallel transmission of b bits over $b + 1$ wires, and admitting especially simple encoding and decoding algorithms. With these codes, resistance to common-mode noise is obtained by using codewords whose components sum to zero, simultaneous switching output noise is reduced by using constant-energy signals, and the effects of intersymbol interference are reduced by having decisions based on only two values at the input of the final slicers. Codebook design is based on the theory of Group Codes for the Gaussian Channel, as specialized to Coxeter matrix groups generated by reflections in orthogonal hyperplanes. A number of designs are exhibited, some of them being novel or improving on previously obtained codes.

Index Terms—Line coding, chordal codes, group codes for the Gaussian channels, permutation modulation, Coxeter groups.

I. INTRODUCTION

IN THIS paper we describe the design of vector line codes allowing an especially simple maximum-likelihood (ML) detection procedure. This consists of a linear transformation of the vector received at the output of an additive white Gaussian noise (AWGN) channel, followed by a binary slicer (a binary hard quantizer). The design is based on the selection of a subset of a permutation modulation (PM) codebook being the direct product of binary antipodal signaling schemes, and hence having a geometrical representation in the form of a multidimensional *orthotope* (or hyper-rectangle). The encoder can also be implemented as a linear transformation of the source (binary) vector.

Transmission on parallel wireline links (as those used to interconnect integrated circuits, or a television set to a set-top box) is affected by disturbances placing a number of constraints on the design of the signaling scheme. The key problem here is the design of line codes allowing the transmission of b bits over $w \geq b$ wires and using a codebook \mathcal{W} subject to some constraints to be detailed later. The general scheme is shown in Fig. 1. Here, b binary information symbols

Manuscript received July 27, 2017; revised February 3, 2018; accepted August 12, 2018. Date of publication September 3, 2018; date of current version February 14, 2019. E. Biglieri was supported by Project TEC2015-66228-P. E. Viterbo was supported by ARC through the Grant Discovery Project under Grant DP160101077. This paper was presented in [3] and [4].

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Communicated by V. Vaishampayan, Associate Editor for Source Coding.

Color versions of one or more of the figures in this paper are available online at <http://ieeexplore.ieee.org>.

Digital Object Identifier 10.1109/TIT.2018.2868453

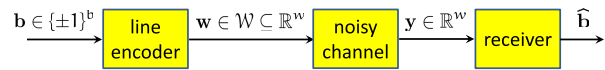
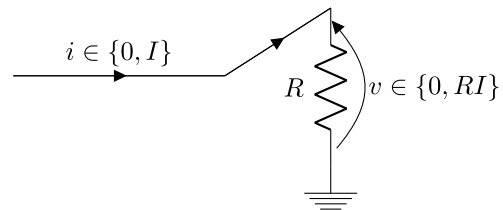
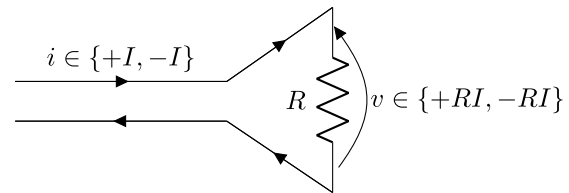


Fig. 1. General scheme of vector coding.



(a)



(b)

Fig. 2. Binary unipolar (NRZ) and binary differential signaling (DS) circuits.

± 1 are input in parallel to the (w, b) line encoder, which is a one-to-one map from $\{\pm 1\}^b$ to the codebook $\mathcal{W} \subseteq \mathbb{R}^w$. This encoder outputs a code vector \mathbf{w} with w real components, which is added to white Gaussian noise to obtain vector \mathbf{y} . Vector \mathbf{y} is processed by a detector whose output is an estimate $\hat{\mathbf{b}}$ of the information vector.

Fig. 2 illustrates the two basic circuits for wired binary communications: (a) unipolar signaling and (b) differential signaling. The transmitter sends a current signal i through one wire in (a) and two wires in (b), and the receiver measures a voltage v across the resistor R . In (a) a threshold voltage $V_t = RI/2$ is used by a comparator to detect the binary information, while in (b) a zero-threshold voltage $V_t = 0$ is used. Since the power dissipated on the resistor R is Ri^2 , on the average (a) uses half of the power of (b), but reduces by a factor of two the distance to the threshold. Since the thermal noise produced by the resistor is the same for both (a) and (b), this implies a power gain of 1.5dB of (b) over (a) to achieve the same performance.

Other types of impairments may heavily affect the reliability of unipolar signaling: (i) Coupling of electromagnetic interference with the transmission wire (common-mode disturbances),

since the return current goes through the ground plane, and (ii) Power supply fluctuations due to simultaneous switching noise (SSN), which affect the stability of the threshold voltage. On the other hand, differential signaling provides common-mode rejection of (i), and the zero threshold is insensitive to (ii) because the total current drawn from the power supply is kept constant (for further details see, for example, [1], [7], and [9]). The advantages of differential signaling (DS) come at the price of a reduction of the wire efficiency from 1 bit/wire to 0.5 bit/wire, and an increase of the complexity needed by the transmitter to drive the currents. Recent work (partially listed among the References below) has focused on the design of signaling schemes that retain the advantages of DS while improving wire efficiency.

The circuit for a (3, 2) line code receiver was illustrated in [3]. In general, the receiver is realized by a star of w resistors with a common center node, where the transmitted zero-sum currents converge to reduce the overall SSN. The w codeword components represent current signals at the transmitter on w wires, and the receiver uses zero-voltage comparators across $\binom{w}{2}$ resistor pairs. These comparators determine the sign of the differences between all pairs of components of the received vector (as illustrated in next Section) in order to provide the sorting order of these components.

In this paper we take an algebraic/geometric approach to the design and analysis of line coding schemes transmitting b bits over $w = b + 1$ wires. Our work was motivated by the invention of *chordal codes* [10]–[12], [21], [25], [26]. With these codes a matrix with orthogonal rows can transform a suitable subset of a PM vector set [27] into a signal constellation whose geometric representation in the Euclidean space is a b -dimensional orthotope, which leads to a simple ML decoding algorithm based on one binary slicer per wire. Code design accounts for several types of impairments that may be present besides additive white Gaussian noise. Specifically, resistance to common-mode noise is obtained by using code words whose components are *balanced* (i.e., sum to zero), simultaneous switching output noise is reduced by using constant-energy signals, and the effects of intersymbol interference are reduced by having only two amplitude values at the input of each slicer [10], [12]. The design method developed in this paper is based on the theory of Group Codes for the Gaussian Channel [28], as specialized to groups generated by reflections in orthogonal hyperplanes (see also [19], [23]).

After examining a simple example for motivation (Section II), a general theory is expounded in Section III. Design examples are shown in Section IV, while performance evaluation is presented in Section V. Section VI deals with the optimization of the design, and additional remarks are presented in Section VII.

II. AN EXAMPLE FOR MOTIVATION AND ILLUSTRATION

An early approach to the code design problem described in previous Section was taken in [21], where a coding scheme based on a number of wires greater than 2 and having a wire efficiency $2/3$ bit/wire was advocated. This scheme was generalized by Abbasfar [1], where a multiwire (“vector”) DS scheme was designed. Under the assumption that the

transmitted amplitudes are ± 1 , the number of $+1$ (and hence of -1) in all transmitted vectors is kept constant, which makes this signaling scheme balanced. An example of this *generalized differential vector signaling* scheme is provided by the following set of 6 vectors (the codebook) used for transmission of $\log_2 6$ bits over $w = 4$ wires. Exhibiting the codebook in the form of a matrix whose rows are the codewords, we have

$$\mathbf{W} = \begin{bmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & +1 & -1 \\ -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 \\ -1 & +1 & -1 & +1 \\ +1 & +1 & -1 & -1 \end{bmatrix} \quad (1)$$

Vectors (1) form a Variant-I PM set [9], [27], [30], obtained as the set of all the permutations of an initial vector $(-1, -1, +1, +1)$. In general, a (Variant-I) PM codebook in \mathbb{R}^w is obtained as the set of all the distinct permutations of an initial w -vector \mathbf{w}_1 . Assuming that \mathbf{w}_1 has r distinct components with multiplicities m_1, \dots, m_r , and $\sum_{i=1}^r m_i = w$, these permutations are in number of $w!/(m_1!m_2!\dots m_r!)$. A peculiar feature of PM is that optimum (ML) detection over the additive white Gaussian noise (AWGN) channel is especially simple. In fact, to decode the received w -vector \mathbf{y} received from a PM codebook, the ML receiver need only arrange its coordinates in decreasing order. This is equivalent to finding the signs of the $\binom{w}{2}$ differences between the components of \mathbf{y} , and comparing these signs with the entries of a lookup table (notice also that the requirement of balanced vectors in the codebook words leads to the optimality of the PM scheme, in the sense discussed in [2]—more on this in Section VI).

Now, line codes based on the PM scheme may be improved upon if a codebook \mathcal{W} can be found such that: (i) It includes a number of codewords equal to a power of 2, so that $|\mathcal{W}| = 2^b$, (ii) Only b signs of linear expressions need be computed for ML detection, (iii) These signs are the source symbols, so that no lookup table is needed by the decoder, and (iv) Encoding can be obtained by a linear operation on source symbols. Line codes satisfying conditions (i)–(iv) were designed by Shokrollahi *et al.* (see [1], [10]–[12], [26], and references within). In this paper we derive a general theory of these codes, based on the concepts of Group Codes for the Gaussian Channel and of Coxeter groups.

We start with a relatively simple design example whose illustration will motivate the theory developed in the balance of this paper. Consider the PM codebook with 6 words¹ obtained as all the permutation of the components of the initial vector $\mathbf{w}_1 = (-1, 0, 1)$. Denoting by i - j the difference between the i th and the j th components of a vector, in the absence of noise the word is identified by the signs of the differences 1-2, 2-3, and 1-3 between pairs of its components. The situation is summarized in Table I, where those differences are shown for all codebook vectors, as received in the absence of noise.

¹In the following, to describe one element of \mathcal{W} we shall use interchangeably the terms word, vector, or point.

TABLE I
VECTORS OF A PM CODEBOOK AND DIFFERENCES
BETWEEN THEIR COMPONENTS

vector	1-2	2-3	1-3
① $(-1, 0, 1)$	-1	-1	-2
② $(-1, 1, 0)$	-2	+1	-1
③ $(0, -1, 1)$	+1	-2	-1
④ $(1, 0, -1)$	+1	+1	+2
⑤ $(0, 1, -1)$	-1	+2	+1
⑥ $(1, -1, 0)$	+2	-1	+1

We can interpret the operations summarized in Table I as a linear mapping \mathcal{L} between the original codebook \mathcal{W} and its transformed version $\mathcal{W}' \triangleq \mathcal{L}\mathcal{W}$, whose words are listed in the right part of Table I. Since this transformation is one-to-one, \mathcal{W}' can be detected in lieu of the original codebook \mathcal{W} . For this observation to be practically useful, we need to consider PM schemes such that decoding \mathcal{W}' in the presence of noise is equivalent to decoding \mathcal{W} , but simpler. The simplest situation, which is the one on which we shall focus our attention in the balance of this paper, occurs when the words of codebook \mathcal{W}' (interpreted as points in the 3-dimensional Euclidean space \mathbb{R}^3) are vertices of a 3-orthotope. If all these vertices are included (which is obtained when the number of codewords chosen is a power of 2, viz., 2^b), then \mathcal{W}' can be optimally detected by simply taking the sign of each entry of the transformed vector $\mathcal{L}\mathbf{y}$, i.e., feeding it to a slicer. The slicer outputs are elements of $\{\pm 1\}^b$. Thus, the ML receiver in this situation consists of a linear transformation \mathcal{L} followed by b slicers. This fact can also be used for *encoding* purposes: in fact, even encoding can be done linearly, by applying a suitable linear transformation to any vector containing, in an appropriate form to be described later, b entries of the form $(\pm 1, \pm 1, \dots, \pm 1)$.²

We now proceed to explain in detail how the concept above can be implemented. The tips of the 6 vectors of Table I are the vertices of a regular hexagon lying on the surface of a 3-dimensional sphere with radius $\sqrt{2}$, as shown in Fig. 3. Since all points of $\mathbf{w} \in \mathcal{W}$ lie on the hyperplane $\langle \mathbf{w}, \mathbf{1} \rangle = 0$, where $\mathbf{1}$ denotes the vector all of whose components are 1, we may project the points of \mathcal{W} on this plane to obtain a 2-dimensional representation. A general way of performing this projection was described by Peterson [24]. The projection of the n -vector \mathbf{w} on the plane described by the scalar product $\langle \mathbf{w}, \mathbf{1} \rangle = 0$ is obtained by computing $\mathbf{w}\mathbf{A}$, where \mathbf{A} is the $n \times n$ projection matrix

$$\mathbf{A} = \begin{bmatrix} 1 + \beta & \beta & \beta & \cdots & \beta & \gamma \\ \beta & 1 + \beta & \beta & \cdots & \beta & \gamma \\ & & \ddots & & & \\ \beta & \beta & \beta & \cdots & 1 + \beta & \gamma \\ \gamma & \gamma & \gamma & \cdots & \gamma & \gamma \end{bmatrix} \quad (2)$$

²It was observed by a reviewer that the motivation for reducing the shape of the signal constellation to an orthotope can also follow from complexity considerations. In fact, the original PM here needs three slicers to transmit $\log_2(6)$ bits, while with the corresponding orthotope only two slicers are needed to transmit $\log_2(4)$ bits. Thus the number of bits per slicer increases from $\log_2(6)/3 \simeq 0.86$ to 1 bits per slicer.

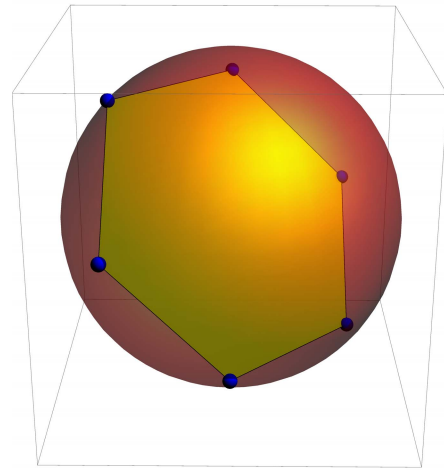


Fig. 3. Geometric representation of codebook in Table I.

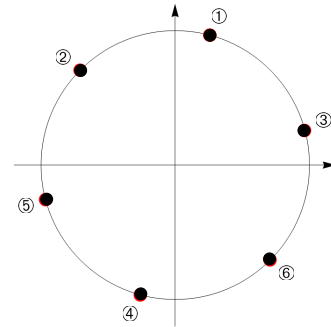


Fig. 4. 2-dimensional geometric representation of codebook in Table I.

where $\gamma \triangleq 1/\sqrt{n}$ and $\beta \triangleq -1/(n - \sqrt{n})$. Using this with $n = 3$, the code vectors are transformed into vectors whose third component is zero, thus reducing the codebook representation to a 2-dimensional space, as illustrated in Fig. 4.

In the specific case we are now handling, the reduced codebook with $2^b = 4$ words whose geometric representation has a rectangular shape is obtained by removing vectors ③ and ⑤ from the 6-vector PM set of Table I. In matrix form:

$$\mathbf{W} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{matrix} \text{①} \\ \text{②} \\ \text{④} \\ \text{⑥} \end{matrix} \quad (3)$$

The ML (congruent) decision regions of this codebook are defined by their boundary planes, as obtained from the equations

$$\langle \mathbf{y}, (\mathbf{w}_i - \mathbf{w}_j) \rangle = 0 \quad (4)$$

where $\mathbf{w}_i, \mathbf{w}_j$ are neighbors. Eq. (4) expresses the fact that the separating plane is orthogonal to the line joining \mathbf{w}_i and \mathbf{w}_j , or, equivalently, that \mathbf{y} has the same distance from \mathbf{w}_i and \mathbf{w}_j . The plane separating neighbors \mathbf{w}_1 and \mathbf{w}_2 has equation $y_2 - y_3 = 0$, while that separating \mathbf{w}_1 and \mathbf{w}_6 has equation $2y_1 - y_2 - y_3 = 0$, or, equivalently, $(y_1 - y_2) + (y_1 - y_3) = 0$ (Fig. 5).

The ML detection procedure is summarized in Table II.

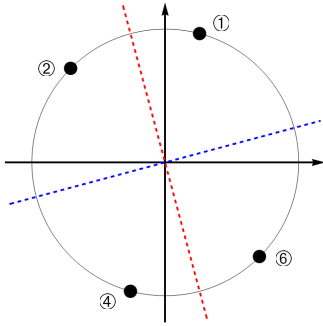


Fig. 5. 2-dimensional representation of codebook (3). Dashed lines: Separators of ML decision regions based on the signs of 2-3 and of (1-2) + (1-3).

TABLE II

TRANSFORMATION OF CODEBOOK \mathbf{W} INTO ONTO THE VECTOR SET \mathbf{W}' WITH 4 ELEMENTS $\pm 3, \pm 1$, THE FOUR VERTICES OF A RECTANGLE

	vector	(1-2) + (1-3)	2-3
①	(-1, 0, 1)	-3	-1
②	(-1, 1, 0)	-3	1
④	(1, 0, -1)	3	1
⑥	(1, -1, 0)	3	-1

This Table describes the transformation \mathcal{L} which maps the codebook \mathbf{W} onto the vector set \mathbf{W}' with 4 elements $\pm 3, \pm 1$, the four vertices of a rectangle. The decision regions of the transformed codebook are delimited by the coordinate axes in the 2-dimensional plane, and hence the transmitted vector can be detected by simply slicing the components of $\mathcal{L}\mathbf{y}$, as indicated above. The linear transformation of \mathbf{W} is given by the *detection matrix*

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad (5)$$

which has orthogonal rows (notice that \mathbf{M} itself is not orthogonal, so that transformation by \mathbf{M} alters the scales of the coordinate axes). Its first row reflects the fact that the sum of the components of each row of \mathbf{W} is zero (balanced codewords), the second row corresponds to the difference (1-2) + (1-3), and the third row to the difference (2-3). Thus, we have

$$\mathbf{WM}^T = \begin{bmatrix} 0 & -3 & -1 \\ 0 & -3 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & -1 \end{bmatrix} \quad (6)$$

which expresses the transformation of the original vector set \mathbf{W} into the new vector set whose two-dimensional representation is shown in Fig. 6. In turn, this vector set can be detected using simply the signs of its second and third components. Conversely, the linear coding procedure transforms the *information matrix*

$$\mathbf{B} = \text{sgn}(\mathbf{WM}^T) = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad (7)$$

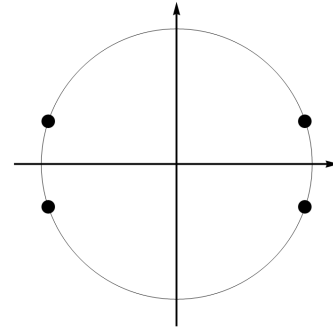


Fig. 6. 2-dimensional representation of the linearly transformed codebook \mathbf{WM}^T .

into \mathbf{W} as

$$\mathbf{W} = \mathbf{BK} \quad (8)$$

where $\mathbf{K} = \frac{1}{2}\mathbf{M}$. Notice that \mathcal{L} transforms the 3-dimensional codebook \mathbf{W} into the 2-dimensional codebook \mathbf{W}' . Thus, the inverse transformation mapping the 4 information symbols into \mathbf{W} should be a map between a 2-dimensional and a 3-dimensional space. This explains why each row of \mathbf{B} has a “zero” prepended.

A. Towards a General Theory

We now describe how codebook (3) can be generated directly, i.e., without necessarily thinking of it as a subset of a PM set. The basic requirement here is that \mathcal{W} be a *group code for the Gaussian channel* [28]. This is generated by the action of a group of real orthogonal matrices \mathbf{O}_i , $i = 1, \dots, 2^b$, on an initial vector \mathbf{w}_1 , and results into Voronoi regions that are congruent. The geometric structure we are interested in is that of an orthotope, which suggests that the Voronoi regions be bounded by orthogonal hyperplanes. In the example we are examining, the two matrices

$$\mathbf{O}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{O}_2 = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \quad (9)$$

satisfy the condition $\mathbf{O}_1^2 = \mathbf{O}_2^2 = \mathbf{I}$, and represent reflections in the orthogonal planes with normal vectors $\mathbf{w}_1 - \mathbf{w}_2 = (0, -1, 1)$ and $\mathbf{w}_1 - \mathbf{w}_6 = (-2, 1, 1)$, respectively. They commute, and $(\mathbf{O}_1\mathbf{O}_2)^2 = \mathbf{I}$. Thus, \mathbf{O}_1 and \mathbf{O}_2 generate the matrix group of order 4 with elements $\mathbf{I}, \mathbf{O}_1, \mathbf{O}_1\mathbf{O}_2, \mathbf{O}_2$. The product of these matrices by the initial vector $(-1, 0, 1)$ yields the codebook (3).

III. BASIC THEORY

Being guided by the considerations developed in Section II, we now expound the general theory leading to line codes in the shape of orthotopes. Our approach, based on finite reflection groups, is chosen as being in the same framework of [20] and [23]. The appropriate mathematical tool is the theory of finite Coxeter groups. These [5], [7], [8], [13], [14] are defined by generators and relations. A finite Coxeter group \mathcal{G} has a presentation with generator set $\mathcal{R} = \{R_i\}$ and

relations $(R_i R_j)^{m(i,j)} = I$, where R_i, R_j are group elements, I denotes the identity element of \mathcal{G} , and $m(i, j)$ are integers. In particular, $m(i, i) = 1$, and $m(i, j) = 2$ if and only if R_i and R_j commute. A Coxeter group has a convenient description in terms of a graph having as nodes the elements of \mathcal{R} and as edges the unordered pairs $\{R_i, R_j\}$ such that $m(i, j) \geq 3$. The edges with $m(i, j) \geq 4$ are labeled by that number. The group is irreducible if its Coxeter graph is connected. For example, the graph with n isolated nodes

$$\bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \quad (10)$$

is the Coxeter graph of a group isomorphic to \mathbb{Z}_2^n of order 2^n . The graph

$$\bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \quad (11)$$

with nodes labeled R_1, R_2, \dots, R_{n-1} , is the Coxeter graph of the symmetric group S_n whose generators are the adjacent transpositions $R_i = (i, i+1)$, $1 \leq i \leq n$.

A reflection on the Euclidean space \mathbb{R}^n is a linear transformation of \mathbb{R}^n of codimension 1, called its *mirror* and having a nontrivial eigenvector with eigenvalue -1 , called a *root* of the reflection. A reflection can be represented by the matrix $\mathbf{I} - 2\delta\delta^T$, where δ is the corresponding unit-norm root vector normal to the mirror plane. Coxeter groups are generated by reflections, so that each node in the Coxeter graph corresponds to a reflection. In particular, when $m(i, j) = 2$ the corresponding reflections are in orthogonal planes. Thus, we are especially interested in groups with Coxeter graphs of the form (10) whose elements are faithfully represented by $(b+1) \times (b+1)$ matrices corresponding to reflections in mutually orthogonal hyperplanes. Once such a group with order 2^b is found, the line code \mathcal{W} is obtained by applying the matrix group to an initial $(b+1)$ -vector \mathbf{w}_1 . As a result, we obtain a line code whose 2^b Voronoi regions are congruent and bounded by orthogonal hyperplanes. In geometric terms, the line code turns out to be equivalent to a Cartesian product of binary antipodal signals, with an added dimension allowing the codewords to be balanced.

To generate from a PM set the Coxeter group we need, we advocate the following procedure. Start from a balanced initial $(b+1)$ -vector \mathbf{w}_1 , and choose b *root permutations*³ \mathbf{w}_i , $i = 2, \dots, b+1$, of \mathbf{w}_1 such that the b unit-norm (column) vectors

$$\delta_i \triangleq \frac{\mathbf{w}_1 - \mathbf{w}_{(i+1)}}{\|\mathbf{w}_1 - \mathbf{w}_{(i+1)}\|}, \quad i = 1, \dots, b \quad (12)$$

are mutually orthogonal. (Section VI describes an algorithm to select these root permutations.) Next, the corresponding reflection matrices

$$\mathbf{O}_i \triangleq \mathbf{I} - 2\delta_i\delta_i^T \quad (13)$$

are computed. Direct calculation shows that $\mathbf{O}_i^2 = (\mathbf{O}_i\mathbf{O}_j)^2 = \mathbf{I}$, so that these matrices generate a Coxeter matrix group isomorphic to a power of \mathbb{Z}_2 . The group code \mathcal{W} is obtained by applying this matrix group to \mathbf{w}_1 [28]. From now

³With an abuse of terminology, we identify a permutation with the vector obtained by permuting the components of \mathbf{w}_1 .

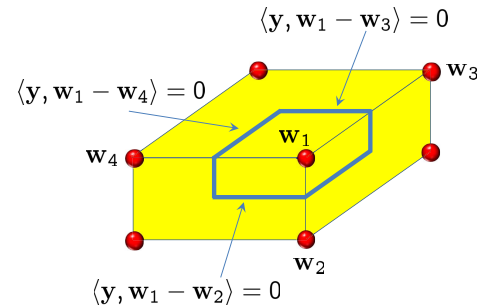


Fig. 7. 3-dimensional Voronoi region for an orthotope-shaped codebook.

on, we shall describe the group code by listing its vectors as rows of the $2^b \times (b+1)$ matrix \mathbf{W} .

Since the Voronoi region associated with a point $\mathbf{w}_i \in \mathcal{W}$ is the set of points lying closer to \mathbf{w}_i than to any other \mathbf{w}_j , $j \neq i$, from the equality $\|\mathbf{y} - \mathbf{w}_i\| = \|\mathbf{y} - \mathbf{w}_j\|$ we see that $\langle \mathbf{y}, (\mathbf{w}_i - \mathbf{w}_j) \rangle = 0$ defines the hyperplane halfway between \mathbf{w}_i and \mathbf{w}_j and orthogonal to the vector $(\mathbf{w}_i - \mathbf{w}_j)$ (see Fig. 7).

The scalar product is positive if \mathbf{y} is closer to \mathbf{w}_i than to \mathbf{w}_j , and vice versa. Thus, defining a matrix \mathbf{M} whose first row is vector $\mathbf{1}$ (reflecting the fact that the whole set of points of \mathcal{W} lies in hyperplane $\langle \mathbf{y}, \mathbf{1} \rangle = 0$) and the remaining rows are proportional to vectors δ_i , the rows of the matrix $\mathbf{W}\mathbf{M}^T$ all have the form $(0, \pm d_1, \dots, \pm d_b)$, with all $d_i > 0$, so that $\text{sgn}\mathbf{W}\mathbf{M}^T = (0, \pm 1, \dots, \pm 1)$. We write

$$\mathbf{W}\mathbf{M}^T = \mathbf{B}\mathbf{D} \quad (14)$$

where \mathbf{D} is the diagonal matrix

$$\mathbf{D} = \text{diag}(0, d_1, \dots, d_b) \quad (15)$$

and the information matrix \mathbf{B} has rows of the form $(0, \pm 1, \dots, \pm 1)$. From (14), we derive the encoding equation $\mathbf{W} = \mathbf{B}\mathbf{K}$, where the encoding matrix \mathbf{K} has the form

$$\mathbf{K} \triangleq \mathbf{D}\mathbf{M}^{-T} \quad (16)$$

where the superscript $-T$ denotes inverse transpose (notice that, since all the entries of the first column of \mathbf{B} are zero, the first row of \mathbf{K} can be replaced by any vector.)

We may summarize our design procedure as follows:

- ① Pick an “initial” vector \mathbf{w}_1 whose $b+1$ real components sum to zero. Considerations on this choice are provided in Section VI-A.
- ② Choose b “root” permutations $\mathbf{w}_2, \dots, \mathbf{w}_{b+1}$ of \mathbf{w}_1 such that the b unit-norm vectors δ_i in (12) are mutually orthogonal. An algorithm to do this is described in Section VI-B.
- ③ Compute the corresponding reflection matrices (13). These are the generators of a Coxeter matrix group \mathcal{G} isomorphic to a power of \mathbb{Z}_2 .
- ④ The code $\mathcal{W} = \mathcal{G}\mathbf{w}_1$ obtained by applying this matrix group to the initial vector \mathbf{w}_1 is a group code, whose codewords are the rows of matrix $\mathbf{W} = \mathbf{B}\mathbf{K}$.

IV. DESIGN EXAMPLES

We shall now exhibit a few design examples, while optimization considerations are postponed to next section. One may notice that we have chosen our operations so that coding and decoding involve only integer numbers. This is not the only choice: for example, one may require all quantities involved in the calculations not to exceed 1 in absolute value. Some of the line codes presented in this Section (Examples 2, 3, 4, and 6) turn out to be equivalent to those designed with different techniques as *chordal codes* in [10]–[12], [25], where we define two codes to be equivalent if they share the same initial vector \mathbf{w}_1 and the same set of performance parameters $\{\alpha_{1,j}\}_{j=2}^{b+1}$ in (40) below. Specifically, our design in Example 3 for $b = 3$ is equivalent to the ENRZ described in [10]), and that in Example 6 for $b = 5$ is equivalent to the CNRZ-5 of [25].

 A. Example 1 ($b = 2$)

Consider the initial vector $\mathbf{w}_1 = (-1, 0, 1)$ and the root permutations $\mathbf{w}_2 = (-1, 1, 0)$ and $\mathbf{w}_3 = (1, -1, 0)$. From (12) we obtain $\delta_1 = (0, -1/\sqrt{2}, 1/\sqrt{2})$ and $\delta_2 = (-2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6})$, the generator matrices (9), and hence codebook (3). With

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & -1 & -1 \end{bmatrix} \quad (17)$$

we obtain

$$\mathbf{WM}^T = \begin{bmatrix} 0 & 1 & -3 \\ 0 & -1 & -3 \\ 0 & -1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \quad (18)$$

which yields $\text{sgn}\mathbf{WM}^T = \mathbf{B}$, where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad (19)$$

With $\mathbf{D} = \text{diag}(0, 1, 3)$ we also obtain the encoding equation

$$\mathbf{BK} = \mathbf{BDM}^{-T} = \mathbf{W} \quad (20)$$

as it should be.

 B. Example 2 ($b = 3$)

With $b = 3$ and initial vector $\mathbf{w}_1 = (-3, -1, 1, 3)$, choose the root permutations $\mathbf{w}_2 = (-3, 3, 1, -1)$, $\mathbf{w}_3 = (-1, -3, 3, 1)$, and $\mathbf{w}_4 = (1, -1, -3, 3)$. From these we obtain $\delta_1 = (0, -1/\sqrt{2}, 0, 1/\sqrt{2})$, $\delta_2 = (-1/2, 1/2, -1/2, 1/2)$, and $\delta_3 = (-1/\sqrt{2}, 0, 1/\sqrt{2}, 0)$, and hence the following generators of the matrix Coxeter group isomorphic to \mathbb{Z}_2^3 :

$$\mathbf{O}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{O}_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{O}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (21)$$

The codebook is

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_1 \mathbf{I} \\ \mathbf{w}_1 \mathbf{O}_1 \\ \mathbf{w}_1 \mathbf{O}_2 \\ \mathbf{w}_1 \mathbf{O}_3 \\ \mathbf{w}_1 \mathbf{O}_1 \mathbf{O}_2 \\ \mathbf{w}_1 \mathbf{O}_1 \mathbf{O}_3 \\ \mathbf{w}_1 \mathbf{O}_2 \mathbf{O}_3 \\ \mathbf{w}_1 \mathbf{O}_1 \mathbf{O}_2 \mathbf{O}_3 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 1 & 3 \\ -3 & 3 & 1 & -1 \\ -1 & -3 & 3 & 1 \\ 1 & -1 & -3 & 3 \\ -1 & 1 & 3 & -3 \\ 1 & 3 & -3 & -1 \\ 3 & -3 & -1 & 1 \\ 3 & 1 & -1 & -3 \end{bmatrix} \quad (22)$$

The Peterson transformation matrix (2)

$$\mathbf{A} = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (23)$$

yields the 3-dimensional version of the codebook

$$\mathbf{AW} = \begin{bmatrix} 0 & 2 & 4 & 0 \\ -4 & 2 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 4 & 2 & 0 & 0 \\ -4 & -2 & 0 & 0 \\ 0 & 2 & -4 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & -2 & -4 & 0 \end{bmatrix} \quad (24)$$

With

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \quad (25)$$

we have that the rows of \mathbf{WM}^T are proportional to $(0, \pm 4, \pm 4, \pm 4)$. Hence, choosing

$$\mathbf{B} = \text{sgn}\mathbf{WM}^T \quad (26)$$

and taking $\mathbf{D} = \text{diag}(0, 4, 4, 4)$, we obtain

$$\mathbf{K} = \mathbf{DM}^{-T} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ -1 & 1 & -1 & 1 \\ -2 & 0 & 2 & 0 \end{bmatrix} \quad (27)$$

and $\mathbf{BK} = \mathbf{W}$, as it should be.

 C. Example 3 ($b = 3$)

Consider again $b = 3$, and the initial vector $\mathbf{w}_1 = (-3, 1, 1, 1)$. Since the permutations of this vector would yield a codebook with only four points, we assume that the central inversion matrix $-\mathbf{I}$ is also an element of the Coxeter group generating the codebook. This is equivalent to assuming that

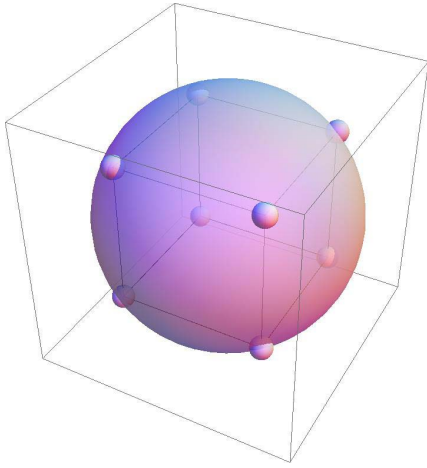


Fig. 8. 3-dimensional representation of the 8 points of codebook of Example 3.

$-\mathbf{w}_1$ is also a codeword. Choosing the root permutations $(-1, 3, -1, -1)$, $(-1, -1, 3, -1)$, and $(-1, -1, -1, 3)$, we obtain the following codebook:

$$\mathbf{W} = \begin{bmatrix} -3 & 1 & 1 & 1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \\ 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \\ 3 & -1 & -1 & -1 \end{bmatrix} \quad (28)$$

which can be seen as the union of two PM codebooks, one generated by the four permutations of $(-3, 1, 1, 1)$ and the other generated by the 4 permutations of $(3, -1, -1, -1)$.⁴

Using (23), the projection $\mathbf{W}\mathbf{A}$ yields a matrix whose rows are the 8 vectors of the form $(\pm 2, \pm 2, \pm 2, 0)$, corresponding to a 3-dimensional cube as shown in Fig. 8.

\mathbf{M} can be given the form of a 4×4 Hadamard matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad (29)$$

which yields a matrix $\mathbf{W}\mathbf{M}^T$ whose rows have the form $(0, \pm 4, \pm 4, \pm 4)$, and the encoding matrix may be chosen as $\mathbf{K} = \mathbf{M}$.⁵

D. Example 4 ($b = 3$)

Choose the initial vector $\mathbf{w}_1 = (-1, 0, 0, 1)$ and the root permutations $(-1, 1, 0, 0)$, $(0, -1, 1, 0)$, and $(0, 0, -1, 1)$.

⁴In a different way, it can be seen as a subset of a Variant-II PM [28], which includes not only the permutations of an initial vector, but also the sign changes of its components. This subset should include only the balanced vectors within the Variant-II PM set.

⁵As pointed out by a reviewer, the detection matrix (25) has fewer nonzero entries than (29), thus making the former simpler to implement. The tradeoff between implementation complexity and performance will not be further commented upon here.

The decoding matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \quad (30)$$

yields rows of the matrix $\mathbf{W}\mathbf{M}^T$ with the form $(0, \pm 1, \pm 2, \pm 1)$. The encoding matrix is $\mathbf{K} = \frac{1}{2}\mathbf{M}$.

E. Example 5 ($b = 4$)

Take $b = 4$, and the initial vector $\mathbf{w}_1 = (-2, -1, 0, 1, 2)$. The root permutations $(-2, 1, 0, -1, 2)$, $(-1, -2, 1, 0, 2)$, $(-1, 0, 1, 2, -2)$, and $(0, -1, -2, 1, 2)$ lead to the decoding matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 4 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (31)$$

which yields rows of the matrix $\mathbf{W}\mathbf{M}^T$ with the form $(0, \pm 2, \pm 2, \pm 10, \pm 4)$.

F. Example 6 ($b = 5$)

With $b = 5$, choose the initial vector $\mathbf{w}_1 = (1, -1, -3, -1, 1, 3)$ and the root permutations $\mathbf{w}_2 = (1, 1, -3, -1, -1, 3)$, $\mathbf{w}_3 = (1, 1, -3, -1, 3, -1)$, $\mathbf{w}_4 = (-1, -1, 1, -3, 1, 3)$, $\mathbf{w}_5 = (-1, -1, -3, 1, 1, 3)$, and $\mathbf{w}_6 = (3, -3, -1, 1, -1, 1)$. The 32-word codebook can be decoded using the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix} \quad (32)$$

The product $\mathbf{W}\mathbf{M}^T$ yields a matrix all of whose rows have the form $(0, \pm 2, \pm 6, \pm 6, \pm 2, \pm 6)$. Encoding is done with $\mathbf{K} = \mathbf{M}$.

V. CONSIDERATION OF ERROR PROBABILITIES

After addition of white Gaussian noise samples $\sim \mathcal{N}(0, N_0/2)$ independent across wires and transmitted b -tuples, the codebook matrix $\mathbf{W} + \mathbf{N}$ is received. The detection process is summarized as the calculation of the signs of $(\mathbf{W} + \mathbf{N})\mathbf{M}^T$. The j th symbol of the i th source b -tuple is erroneously detected if its polarity is altered by noise, which occurs with probability

$$(p_e)_{i,j} = \mathbb{P} \left(n_j < - \left| (\mathbf{W}\mathbf{M}^T)_{i,j} \right| \right) \quad (33)$$

where $n_j \sim \mathcal{N}(0, \sigma_j^2)$, and $\sigma_j^2 \triangleq (N_0/2)\xi_j^2$ is the j th element of the diagonal covariance matrix of the noise term:

$$\begin{aligned} \mathbb{E} \left[(\mathbf{N}\mathbf{M}^T)^T (\mathbf{N}\mathbf{M}^T) \right] &= \mathbf{M} \left[\mathbb{E} (\mathbf{N}^T \mathbf{N}) \right] \mathbf{M}^T \\ &= \frac{N_0}{2} \text{diag}(\xi_1^2, \dots, \xi_{b+1}^2) \end{aligned} \quad (34)$$

Thus,

$$(p_e)_{i,j} = Q\left(\frac{\left|(\mathbf{W}\mathbf{M}^T)_{i,j}\right|}{\sqrt{N_0/2}\xi_j}\right) \quad (35)$$

We define the signal-to-noise ratio η observing that the average energy associated with the transmission of a signal b -tuple is given by

$$\mathcal{E} = \frac{\|\mathbf{W}\|^2}{2^b} \quad (36)$$

where $\|\mathbf{W}\|$ denotes the Frobenius norm of matrix \mathbf{W} . The energy per bit is consequently $\mathcal{E}_b = \mathcal{E}/b$, and the signal-to-noise ratio is

$$\eta \triangleq \frac{\mathcal{E}_b}{N_0} = \frac{\|\mathbf{W}\|^2/2^b}{bN_0} \quad (37)$$

Thus, we can rewrite (35) in the form

$$(p_e)_{i,j} = Q\left(\alpha_{i,j}\sqrt{2\eta}\right), \quad i = 1, \dots, 2^b, \quad j = 2, \dots, b+1 \quad (38)$$

where⁶

$$\alpha_{i,j} \triangleq \frac{\left|(\mathbf{W}\mathbf{M}^T | (\mathbf{M}\mathbf{M}^T)^{-1/2})_{i,j}\right|}{\sqrt{\|\mathbf{W}\|^2/(b2^b)}} \quad (39)$$

Since group codes have the *uniform error probability*, i.e., the error probability is the same for every transmitted code-word, the values of $\alpha_{i,j}$ do not depend on the value of i . An alternative expression is

$$\alpha_{1,j} = \sqrt{b} \frac{\|\mathbf{w}_1 - \mathbf{w}_j\|}{2 \cdot \|\mathbf{w}_1\|}, \quad j = 2, \dots, b+1, \quad (40)$$

where the denominator is the diameter of the sphere enclosing the codebook vectors.

Using (40), we obtain

$$\sum_{j=2}^{b+1} \alpha_{1,j}^2 = b \quad (41)$$

In fact, due to the symmetry of the codebook, if $\mathbf{w}_1 \in \mathcal{W}$ then also $-\mathbf{w}_1 \in \mathcal{W}$. The distance between \mathbf{w}_1 and $-\mathbf{w}_1$ is equal to the diameter of the hypersphere on whose surface the codebook points lie, and the vector joining these two points is the longest diagonal of the corresponding orthotope. The squared length of this diagonal equals the sum of the squared lengths of the edges radiating from \mathbf{w}_1 , which proves (41).

Since the Voronoi regions are congruent and bounded by orthogonal hyperplanes, the ML decisions on the individual bits are affected by independent noise samples, and hence the

⁶Observe that the rows of the matrix $|\mathbf{W}\mathbf{M}^T|$ quantifies the amplitudes of the eye opening before rectification. Notice that having equal columns of matrix $|\mathbf{W}\mathbf{M}^T|$ is not sufficient to have equally protected symbols, as their noise protection also depends on the values $\xi_2^2, \dots, \xi_{(b+1)}^2$.

following exact expression for the average error probability holds:

$$p_e = 1 - \prod_{j=2}^{b+1} \left[1 - Q\left(\alpha_{1,j}\sqrt{2\eta}\right)\right] \quad (42)$$

We may also observe that (42) is minimized, under the constraint (41), by choosing all the $\alpha_{1,j}$ equal, which corresponds to having the codebook orthotope equal to a hypercube.

From (42) we may derive the union upper bound

$$p_e \leq \sum_{j=2}^{b+1} Q\left(\alpha_{1,j}\sqrt{2\eta}\right) \quad (43)$$

and the asymptotic approximation, valid for large signal-to-noise ratios,

$$p_e \lesssim \nu Q\left(\alpha_{\min}\sqrt{2\eta}\right) \quad (44)$$

where $\alpha_{\min} = \min_j \alpha_{1,j}$, and ν is the number of $\alpha_{1,j}$ taking value α_{\min} .

Table III summarizes the values of the $\alpha_{i,j}$ for some line codes.

VI. OPTIMIZATION OF THE CODEBOOK

A natural and common optimization criterion, based on the performance at large values of signal-to-noise ratios, is the maximization of the minimum Euclidean distance of the codebook, i.e., of

$$d_{\min} = \min_{\mathbf{O}} \|\mathbf{w}_1 - \mathbf{O}\mathbf{w}_1\| \quad (45)$$

where \mathbf{O} runs through the matrices representing the Coxeter group chosen for the codebook design. The choice between two line codes with the same d_{\min} may be based on the second smallest Euclidean distance, etc. Since the design criterion described in Section III generates a codebook which is a subset of a PM set, the minimum distance of the latter turns out to be a lower bound on (45).

A. Choosing the Initial Vector

The first constraint on the choice of \mathbf{w}_1 comes from the observation that, due to the linearity of the encoder, if $\mathbf{w} \in \mathcal{W}$ then also $-\mathbf{w} \in \mathcal{W}$. A sufficient condition for this to occur is to force $-\mathbf{w}$ to be a permutation of \mathbf{w} , which is obtained from an initial vector such that its nonzero components occur in pairs including positive and negative values. All the examples in Table III satisfy this condition, with the only exception of the entry described in Example IV-C.

Further, it seems reasonable to start from an original PM set having the largest possible minimum distance.⁷ Using the notations of [27], the initial vector for the generation of a PM set has components μ_1, \dots, μ_k , each being different and repeated m_1, \dots, m_k times, respectively. It was proved in [2] that for optimality the μ_i must be equally spaced (i.e., $\mu_{i+1} - \mu_i$ is a constant). Moreover, if m_1, \dots, m_k are given, then the optimum combination of μ s and m s consists of

⁷One should keep in mind that it may occur that the minimum distance of the line code be larger than that of the original PM set.

TABLE III
PERFORMANCE OF SOME $(b + 1, b)$ COXETER-GROUP LINE CODES.

b	\mathbf{w}_1	root permutations	$\alpha_{1,j}, j = 2, \dots, b + 1$
1	(1, -1)	(-1, 1)	1.
2	(-1, 0, 1)	(-1, 1, 0) (1, -1, 0)	0.71, 1.22
3	(-3, -1, 1, 3)	(-3, 3, 1, -1) (-1, -3, 3, 1) (1, -1, -3, 3)	0.77, 1.1, 1.1
3	(-1, 0, 0, 1)	(-1, 1, 0, 0) (0, -1, 1, 0) (0, 0, -1, 1)	0.87, 0.87, 1.22
3	(-3, 1, 1, 1)	(-1, 3, -1, -1) (-1, -1, 3, -1) (-1, -1, -1, 3)	1, 1, 1
4	(-2, -1, 0, 1, 2)	(-2, 1, 0, -1, 2) (-1, -2, 1, 0, 2) (-1, 0, 1, 2, -2) (0, -1, -2, 1, 2)	0.63, 0.89, 0.89, 1.41
5	(1, -1, 3, -3, 5, -5)	(-1, 1, 5, -5, 3, -3) (3, -3, 1, -5, 5, -1) (3, -3, 5, -1, 1, -5) (3, 5, -3, -1, 1, -5) (-5, -3, 1, 3, 5, -1)	0.66, 0.76, 0.76, 1.31, 1.31
5	(-2, -1, 0, 0, 1, 2)	(-2, -1, 2, 0, 1, 0) (-2, 0, -1, 0, 2, 1) (-2, 1, 0, 0, -1, 2) (0, -2, -1, 2, 0, 1) (0, -1, 0, -2, 1, 2)	0.71, 1.0, 1.0, 1.0, 1.22
5	(1, -1, -3, -1, 1, 3)	(1, 1, -3, -1, -1, 3) (1, 1, -3, -1, 3, -1) (-1, -1, 1, -3, 1, 3) (-1, -1, -3, 1, 1, 3) (3, -3, -1, 1, -1, 1)	0.67, 0.67, 1.17, 1.17, 1.17

pairing the smallest m with the smallest μ , the second smallest m with the largest μ , the third smallest m with the second smallest μ , and so forth. Thus, the optimization of a PM set is complete once the m s are chosen in an optimum way. Ingemarsson [15], [16] has advocated a choice of the m s which makes the amplitudes of the initial vector have a sampled Gaussian distribution (an idea that was used in [27]). However, the solution of [15] and [16] may not be optimum, as the search was restricted to initial vectors satisfying a certain symmetry [19], [20]. A numerical optimization algorithm was derived by Karlof [17] and Karlof and Chang [18], while tables of optimum PM sets in low dimensions are exhibited in [9]. For small values of b , a sensible choice consists of checking all the partitions (m_1, m_2, \dots, m_r) of the number of components of \mathbf{w}_1 , as mentioned in Section II, and choosing the partition yielding the best code.

In our design, for d_{\min} optimization, we examine all the partitions of $(b + 1)$ in the form $b + 1 = m_1 + \dots + m_k$, and derive for each of them a codebook under the assumption of equally spaced μ_i . This is shown in Examples IV-B to IV-D, where the partitions $4 = 1 + 1 + 1 + 1$, $4 = 1 + 2 + 1$, and $4 = 3 + 1$ were considered. Notice also that some of the partitions may not lead to a codebook satisfying our constraints: for example, the partition $4 = 2 + 2$ generates a PM set with

$4!/(2!2!) = 6$ vectors, which cannot be used to generate a codebook with $2^b = 8$ vectors as needed.

B. Choosing the Root Permutations

Once \mathbf{w}_1 has been chosen, the Coxeter matrix group has to be generated, which is obtained, as described in Section III, by taking b additional permutations $\mathbf{w}_i, i = 2, \dots, b + 1$, (the root permutations) such that the b difference vectors $(\mathbf{w}_1 - \mathbf{w}_i)$ are mutually orthogonal. These vectors correspond to the b edges of a b -dimensional orthotope having \mathbf{w}_1 as a vertex. We observe first that in some cases such permutations may not exist. For example, the initial vector $\mathbf{w}_1 = (-1, 0, 0, 0, 1)$ originates a PM set with 20 vectors, from which 4 orthogonal difference vectors cannot be found. In other cases, more than one choice of root permutations is available, as shown graphically in Fig. 9 for the simple case $b = 2$. From this figure it is seen that the two choices are equivalent, as they give rise to congruent orthotopes, but this may not be the case for $b > 2$.

In other cases the different choices of root permutations yield codebooks with different performance, as revealed by the Euclidean distances from \mathbf{w}_1 to the other root permutations. A simple algorithm listing all the choices of root permutations

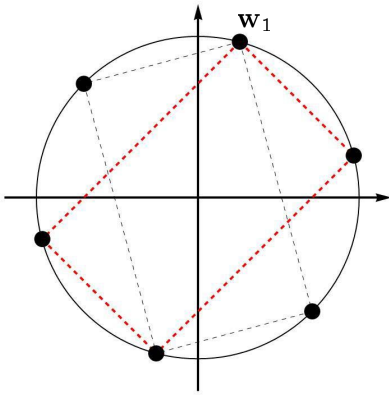


Fig. 9. Two possible choices of root permutations leading to equivalent codebooks with $b = 2$.

and their quality consists of the following. Consider the graph whose vertices are the differences between \mathbf{w}_1 and all its permutations, \mathbf{x}' and whose edges join the vertices corresponding to orthogonal differences. Any complete subgraph where each vertex is adjacent to every other (called a *clique*) yields a set of mutually orthogonal differences. The maximum number of vertices in such a clique is b , and the clique is called maximal. Thus, the choice of a set of root permutations is tantamount to the choice of a maximal clique with the largest minimum norm of the orthogonal differences in it (as mentioned before, if two cliques have the same minimum norm, we choose the one whose second smallest distance is the largest, etc.). To generate the graph, form the matrix Δ whose rows are the differences between \mathbf{w}_1 and all its permutations. The Gram matrix $\Delta\Delta^T$ has a zero in all entries corresponding to a pair of orthogonal differences. Replacing the zero entries in $\Delta\Delta^T$ with 1's and the nonzero entries with 0's yields the incidence matrix of the graph sought. Algorithms (see [6]) are available to list the maximal cliques.

For example, the initial vector $\mathbf{w}_1 = (1, -1, -3, -1, 1, 3)$ of Example IV-F has 180 permutations and 24 maximal cliques. The best clique under our criterion yields the values of $\alpha_{1,j}$ listed in the last entry of Table III. A related line code, using the same \mathbf{w}_1 and exhibited in [26, Table 2], yields a slightly inferior performance (the values of $\alpha_{1,j}$ are 0.67, 0.67, 0.95, and 1.65).

C. Removing the PM and Integer-Number Constraint

The designs done in the previous sections were based on the constraint of a codebook being a subset of a PM set, as this choice reduces the cardinality of the set of the signal amplitudes in each wire. This is a convenient choice, because a limited number of amplitudes implies a limited number of current or voltage sources needed to implement the encoder. In addition, one can deal only with integer amplitudes, thus increasing the accuracy of the implementation as rounding becomes unnecessary. The downside of this choice is that the vertices of the codebook orthotope are constrained to a subset of those of the polytope of the original PM set (a *semiregular* polytope, see [27]). If this constraint is removed, after a Coxeter matrix group is generated, one may choose the optimum initial vector as indicated in [20], that is, being at

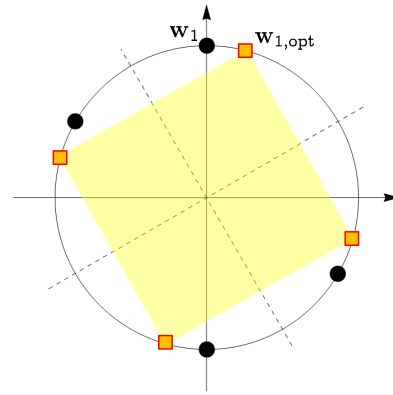


Fig. 10. Comparison of the line codes with $b = 2$ obtained from the same matrix Coxeter group, as applied to \mathbf{w}_1 (see the entry with $b = 2$ in Table III) and to $\mathbf{w}_{1,\text{opt}}$.

the same distance from every plane bounding the fundamental region of the Coxeter group in which \mathbf{w}_1 lies.⁸ In general, we have

$$\mathbf{w}_{1,\text{opt}} = \sum \frac{\mathbf{w}_1 - \mathbf{w}_i}{\|\mathbf{w}_1 - \mathbf{w}_i\|} \quad (46)$$

where the sum runs through the set of root permutations. The resulting codebook has the shape of a hypercube, which yields $\alpha_{1,j} = 1$ for $j = 2, \dots, b + 1$. Fig. 10 shows a geometric representation of the codebook obtained with the choice of the optimum initial vector.

For suboptimum design, one may use an integer-value approximation of the optimum initial vector, which could lead to a codebook shape close to a hypercube, possibly at the price of a larger number of values of the codeword components. For illustration, consider $b = 3$ and the first set of root permutations of Table III. The matrix representation of the Coxeter group in this case yields the optimum initial vector $\mathbf{w}_{1,\text{opt}} = (-1/2 - \sqrt{2}/2, 1/2 - \sqrt{2}/2, -1/2 + \sqrt{2}/2, 1/2 + \sqrt{2}/2)$, and a codebook \mathbf{W} whose rows are permutations of $\mathbf{w}_{1,\text{opt}}$. Using the suboptimum initial vector $(-6, -1, 1, 6) \approx 5 \times \mathbf{w}_{1,\text{opt}}$, one obtains $\alpha_{1,j} \approx 1$.

VII. MISCELLANEOUS REMARKS

- ① In [20] and [23], group codes generated by Coxeter groups were studied. The constraint of having Voronoi regions bounded by orthogonal hyperplanes, and hence allowing an exceedingly simple ML detection, was not considered. The designs in [20] were optimized by choosing an initial vector in the center of a fundamental region of the Coxeter group.
- ② A topic related to the codebooks examined in this paper is the study of constant-weight codes. These satisfy the equal-energy condition, while their words may not be balanced in the sense of this paper. See [22], [29], and the references therein.
- ③ Introduction of error-control capabilities can be obtained by suitably decreasing the wire efficiency and using standard linear codes, as advocated in [12].

⁸We recall that the fundamental region of a matrix group is a connected region of the space such that no point in its interior can be obtained as $\mathbf{O}\mathbf{w}_1$, where \mathbf{O} is any matrix of the group. For a precise definition see [20].

- ④ It should be noticed that the subset of permutations leading to a line code constructed using a Coxeter group does not necessarily form a subgroup of matrices of the natural representation of the symmetric group. In fact, although the generating matrices of a representation of the Coxeter group, as applied to \mathbf{w}_1 , yield permutations of that vector, these may not be all permutation matrices. As an example, the “square” code of Fig. 10, generated by the optimum \mathbf{w}_1 approximately equal to $(-0.8, -0.3, 1.1)$, yields the codebook

$$\mathbf{W} = \begin{bmatrix} -0.8 & -0.3 & 1.1 \\ -0.8 & 1.1 & -0.3 \\ 0.8 & -1.1 & 0.3 \\ 0.8 & 0.3 & -1.1 \end{bmatrix} \quad (47)$$

As another example, with the design of last entry in Table III the optimum initial vector can be found to be $(1, -1, -\sqrt{3}, -1, 1, \sqrt{3})$ yields a codebook whose words have again an increased alphabet size and are not permutations of the initial vector.

VIII. CONCLUSIONS

Expanding on the work described in [1], [10], [12], and [26], we have developed an algebraic method for generating line codes for parallel transmission that have many of the properties of differential signaling. These are group codes generated by a matrix representation of a Coxeter group. Performance evaluation is also discussed, and a number of design examples are exhibited (some of which are new, or improve upon known codes), along with some consideration of optimum codes.

ACKNOWLEDGMENT

The authors gratefully acknowledge the comments of Michele Elia on an early version of this paper. They would like to thank also the Associate Editor Vinay Vaishampayan and the anonymous reviewers for their beneficial comments, which improved the quality of this paper.

REFERENCES

- [1] A. Abbasfar, “Generalized differential vector signaling,” in *Proc. IEEE Int. Conf. Commun. (ICC)*, Dresden, Germany, Jun. 2009, pp. 1–5.
- [2] E. Biglieri and M. Elia, “Optimum permutation modulation codes and their asymptotic performance,” *IEEE Trans. Inf. Theory*, vol. IT-22, no. 6, pp. 751–753, Nov. 1976.
- [3] E. Biglieri and E. Viterbo, “Line coding for differential vector signaling,” in *Proc. ITA Workshop*, San Diego, CA, USA, Feb. 2017, pp. 1–6.
- [4] E. Biglieri and E. Viterbo, “Geometrically uniform differential vector signaling schemes,” in *Proc. IEEE ISIT*, Aachen, Germany, Jun. 2017, pp. 506–510.
- [5] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*. New York, NY, USA: Springer, 2005.
- [6] C. Bron and J. Kerbosch, “Algorithm 457: Finding all cliques of an undirected graph,” *Commun. ACM*, vol. 16, no. 9, pp. 575–577, 1973.
- [7] H. S. M. Coxeter, “Discrete groups generated by reflections,” *Am. Math.*, vol. 35, no. 3, pp. 588–621, 1934.
- [8] H. S. M. Coxeter, *Regular Polytopes*. London, U.K.: Methuen, 1948.
- [9] T. Ericson, “Permutation codes,” INRIA, Paris, France, Res. Rep. 2109, Nov. 1993.
- [10] (Aug. 2016). *A Primer on Chord Signaling*. [Online]. Available: <https://www.kandou.com/technology/coding>
- [11] H. Cronie and A. Shokrollahi, “Orthogonal differential vector signaling,” U.S. Patent Appl. 2011/0268225 A1, Nov. 3, 2011.
- [12] H. Cronie and A. Shokrollahi, “Power and pin efficient chip-to-chip communications with common-mode rejection and SSO resilience,” U.S. Patent Appl. 2011/0302478 A1, Dec. 8, 2011.
- [13] L. C. Grove and C. T. Benson, *Finite Reflection Groups*. New York, NY, USA: Springer, 1985.
- [14] J. E. Humphreys, *Reflection Groups and Coxeter Groups*. Cambridge, U.K.: Cambridge Univ. Press, 1985.
- [15] I. Ingemarsson, “Optimized permutation modulation,” *IEEE Trans. Inf. Theory*, vol. 36, no. 5, pp. 1098–1100, Sep. 1990.
- [16] I. Ingemarsson *et al.*, “Group codes for the Gaussian channel,” in *Topics in Coding Theory* (Lecture Notes in Control and Information Sciences), vol. 128, G. Einarsson, Ed. New York, NY, USA: Springer-Verlag, 1989, pp. 73–108.
- [17] J. Karlof, “Permutation codes for the Gaussian channel,” *IEEE Trans. Inf. Theory*, vol. 35, no. 4, pp. 726–732, Jul. 1989.
- [18] J. K. Karlof and Y. O. Chang, “Optimal permutation codes for the Gaussian channel,” *IEEE Trans. Inf. Theory*, vol. 43, no. 1, pp. 356–358, Jan. 1997.
- [19] T. Mittelholzer, “Construction and decoding of optimal group codes from finite reflection groups,” in *Communications and Cryptography: Two Sides of One Tapestry*, R. E. Blahut *et al.*, Eds. Norwell, MA, USA: Kluwer, 1994.
- [20] T. Mittelholzer and J. Lahtonen, “Group codes generated by finite reflection groups,” *IEEE Trans. Inf. Theory*, vol. 42, no. 2, pp. 519–528, Mar. 1996.
- [21] D. Oh *et al.*, “Pseudo-differential signaling scheme based on 4b/6b multiwire code,” in *Proc. IEEE 17th Top. Meeting Elect. Perform. Electron. Packag. (EPEP)*, San Jose, CA, USA, Oct. 2008, pp. 29–32.
- [22] D. Pelusi, S. Elmougy, L. G. Tallini, and B. Bose, “*m*-ary balanced codes with parallel decoding,” *IEEE Trans. Inf. Theory*, vol. 61, no. 6, pp. 3251–3264, Jun. 2015.
- [23] W. W. Peterson, J. B. Nation, and M. P. Fossorier, “Reflection group codes and their decoding,” *IEEE Trans. Inf. Theory*, vol. 56, no. 12, pp. 6273–6293, Dec. 2010.
- [24] W. W. Peterson, “A note on permutation modulation,” *IEEE Trans. Inf. Theory*, vol. 43, no. 1, pp. 359–360, Jan. 1997.
- [25] A. Shokrollahi *et al.*, “A pin-efficient 20.83Gb/s/wire 0.94pJ/bit forward clock CNRZ-5-coded SerDes up to 12mm for MCM packages in 28nm CMOS,” in *IEEE ISSCC Dig. Tech. Papers*, San Francisco, CA, USA, Jan./Feb. 2016, pp. 182–183.
- [26] A. Shokrollahi and R. Ulrich, “Vector signaling codes with increased signal to noise characteristics,” U.S. Patent Appl. 2016/0013954 A1, Jan. 14, 2016.
- [27] D. Slepian, “Permutation modulation,” *IEEE Proc.*, vol. 53, no. 3, pp. 228–236, Mar. 1965.
- [28] D. Slepian, “Group codes for the Gaussian channel,” *Bell Syst. Tech. J.*, vol. 47, no. 4, pp. 575–602, Apr. 1968.
- [29] L. Tallini and B. Bose, “Transmission time analysis for the parallel asynchronous communication scheme,” *IEEE Trans. Comput.*, vol. 52, no. 5, pp. 558–571, May 2003.
- [30] E. Viterbo, “Permutation codes,” in *Encyclopedia of Telecommunications*, J. Proakis, Ed. Hoboken, NJ, USA: Wiley, 2002.

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