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OTFS Performance on Static Multipath Channels

P. Raviteja, Emanuele Viterbo, and Yi Hong

***Abstract*—Orthogonal time frequency space (OTFS) modulation is known to achieve excellent error performance in delay–Doppler channels. However, its performance over static multipath channels has never been fully investigated. In this letter, we show that, in static multipath channels, the system structure of OTFS is equivalent to the asymmetric orthogonal frequency division multiplexing (A-OFDM), a scheme proposed in [1], bridging between cyclic prefix single carrier (CPSC) and traditional OFDM. We derive a condition on the parameters of OTFS to guarantee that all the transmitted symbols experience uniform channel gains, as in CPSC. Finally, we apply a low-complexity message passing detection to OTFS/A-OFDM and show a significant performance improvement over ZF and MMSE detection originally proposed for A-OFDM.**

***Index Terms*—OTFS, OFDM, static multipath channels, message passing**

I. Introduction

Recently proposed orthogonal time frequency space (OTFS) modulation offers great potential to handle multiple high Doppler shifts in time-varying multipath wireless channels [3]–[5]. The key idea of OTFS is to transmit information symbols in the delay–Doppler domain to better resolve the delay and Doppler multiple paths of a time-varying wireless channel. The information symbols placed in delay-Doppler domain can be transformed to the standard time-frequency domain, used in traditional modulation schemes such as orthogonal frequency division multiplexing (OFDM). Such transformation is realized via a set of two-dimensional (2D) bi-orthogonal basis functions shaped by ideal pulse-shaping waveforms. As a result, all information symbols experience a constant flat fading equivalent channel [3].

Since the first paper on OTFS in [3], a number of system improvements were proposed in [5]–[18]. All these works have studied OTFS over time-variant (high Doppler) wireless channels, yet the complete analysis of OTFS over time-invariant (static) multipath channels has never been explored.

In this letter, we study OTFS over static multipath channels and reveal that the system structure of OTFS is equivalent to the asymmetric orthogonal frequency division multiplexing (A-OFDM), a scheme proposed in [1] that generalizes OFDM and cyclic prefix single carrier (CPSC) by exploiting a layered FFT structure. Next, we derive a necessary and sufficient condition on the number of subcarriers in OTFS to guarantee that all the transmitted symbols experience uniform channel gains, as in CPSC (a special case of OTFS/A-OFDM). We also show that OTFS offers a tradeoff between spectral efficiency and maximum peak-to-average power ratio (PAPR), for a given performance target and detection complexity. Finally, we apply a message passing (MP) algorithm to OTFS/A-OFDM and show that it outperforms OFDM as well as A-OFDM with ZF and MMSE detectors in [1].

Notations: We denote scalar, vector, and matrix by a, $\mathbf{a}$, and $\mathbf{A}$, respectively. We let $\mathbf{a}(i)$ and $\mathbf{A}(i,j)$ represent the i^{th} element of $\mathbf{a}$ and $(i,j)^{th}$ element of $\mathbf{A}$. We denote the set of $M \times N$ dimensional matrices with each entry from the complex plane by $\mathbb{C}^{M\times N}$. Let $\mathbf{A} = \text{circ}[\mathbf{A}_0, \cdots, \mathbf{A}_{N-1}]$ and $\text{diag}[\mathbf{A}_0, \cdots, \mathbf{A}_{N-1}] \in \mathbb{C}^{MN\times MN}$ represent the block circulant matrix with first column block as $\{\mathbf{A}_0, \cdots, \mathbf{A}_{N-1}\} \in \mathbb{C}^{M\times M}$, and the block diagonal matrix with $\{\mathbf{A}_0, \cdots, \mathbf{A}_{N-1}\}$ as diagonal blocks, respectively. We let superscript H denote Hermitian transposition, the $\otimes$ operation denote Kronecker product, and the $\text{vec}(\cdot)$ operation denote the column vectorization of an $M \times N$ (complex) matrix into an $MN \times 1$ (complex) column vector. Finally, we let $\mathbf{F}_n = \{\frac{1}{\sqrt{n}} e^{2\pi jkl/n}\}_{k,l=0}^{n-1}$ and $\mathbf{F}_n^{\rm H}$ be the n-point DFT and the IDFT matrices, and the term $\mathbf{I}_M$ be a M-dimensional identity matrix.

II. OTFS in Static Multipath Channels

We consider an OTFS system with single antenna transmitter and receiver over static multipath channels, i.e., the channel consists of P zero-Doppler multipaths with the i^{th} path delay denoted by τ_i, for $i = 1, 2, \cdots, P$. We assume that a total of $N_c = MN$ symbols are transmitted in an OTFS frame of duration $N_c T_s$, where T_s is the sampling interval. Let $\tau_{\max} = (L-1)T_s$ denote the maximum delay of an L-tap channel. The static multipath channel is represented by the L tap coefficients $[h_0, h_1, \cdots, h_{L-1}]$, where only P elements are non-zero.

Let $\mathbf{x} = \text{vec}(\mathbf{X}) \in \mathbb{C}^{N_c\times 1}$ denote one OTFS frame containing N_c transmitted information symbols, each with average energy E_s, where the matrix $\mathbf{X} \in \mathbb{C}^{M\times N}$ represents the two-dimensional information symbols transmitted in the delay-Doppler plane. The transmitted time domain signal in OTFS can be obtained by first applying the (2D) *ISFFT* on $\mathbf{X}$ followed by *Heisenberg transform* [3]. Assuming rectangular transmit waveform, the output of the Heisenberg transform can be written as [8]

$$\mathbf{S} = \mathbf{F}_M^{\rm H}(\mathbf{F}_M \mathbf{X} \mathbf{F}_N^{\rm H}) = \mathbf{X}\mathbf{F}_N^{\rm H} \quad (1)$$

The transmitted time domain signal can be generated by column-wise vectorization of $\mathbf{S}$:

$$\mathbf{s} = \text{vec}(\mathbf{S}) = (\mathbf{F}_N^{\rm H} \otimes \mathbf{I}_M)\mathbf{x} \quad (2)$$

We assume a CP of length $(L-1)$ is added to $\mathbf{s}$ before transmission. The received signal in time domain, after discarding the CP, can be written as

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{w}, \quad (3)$$

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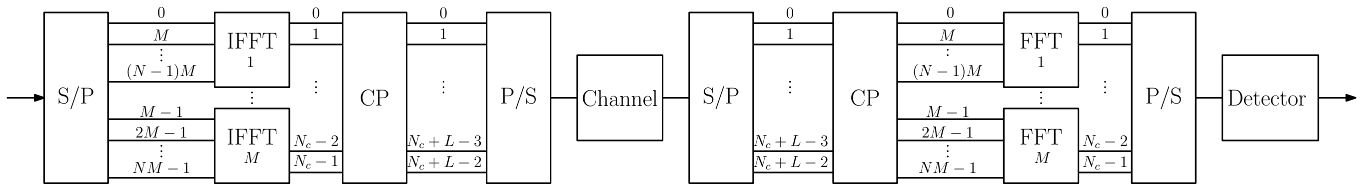

Fig. 1. OTFS/A-OFDM for static multipath channels

where $\mathbf{H} = \text{circ}[h_0, h_1, \cdots, h_{L-1}, 0, \cdots, 0] \in \mathbb{C}^{N_c \times N_c}$ is the circulant matrix, and $\mathbf{w} \in \mathbb{C}^{N_c \times 1}$ is the i.i.d. Gaussian noise vector with the i^{th} entry, $w_i \sim \mathcal{CN}(0, \sigma^2)$.

At the receiver, the received signal $\mathbf{r}$ is devectorized into an $M \times N$ matrix $\mathbf{R}$, followed by a *Wigner transform* as well as a SFFT, yielding

$$\mathbf{Y} = \mathbf{F}_M^{\mathrm{H}}(\mathbf{F}_M \mathbf{R})\mathbf{F}_N = \mathbf{R}\mathbf{F}_N \tag{4}$$

Finally, the input–output relation of OTFS in the information domain can be obtained by column-wise vectorization of (4):

$$\begin{aligned} \mathbf{y} &= \text{vec}(\mathbf{Y}) = (\mathbf{F}_N \otimes \mathbf{I}_M)\mathbf{r} \\ &= (\mathbf{F}_N \otimes \mathbf{I}_M)\mathbf{H}(\mathbf{F}_N^{\mathrm{H}} \otimes \mathbf{I}_M)\mathbf{x} + \widetilde{\mathbf{w}} \\ &= \mathbf{H}_{\text{eff}}\mathbf{x} + \widetilde{\mathbf{w}} \end{aligned} \tag{5}$$

where $\mathbf{H}_{\text{eff}} = (\mathbf{F}_N \otimes \mathbf{I}_M)\mathbf{H}(\mathbf{F}_N^{\mathrm{H}} \otimes \mathbf{I}_M)$ is the effective channel matrix. Since $(\mathbf{F}_N \otimes \mathbf{I}_M)$ is a unitary matrix, $\widetilde{\mathbf{w}} = (\mathbf{F}_N \otimes \mathbf{I}_M)\mathbf{w}$ preserves the same statistical properties of $\mathbf{w}$.

Let us consider the following two special cases for OTFS over static multipath channels.

1) If $M = 1$ then (4) reduces to $\mathbf{y} = \mathbf{F}_N \mathbf{H} \mathbf{F}_N^{\mathrm{H}} \mathbf{x} + \widetilde{\mathbf{w}}$, i.e., a conventional N-subcarrier OFDM system, when a CP is added to $\mathbf{x}$.
2) If $N = 1$ then (4) reduces to $\mathbf{y} = \mathbf{H}\mathbf{x} + \widetilde{\mathbf{w}}$, i.e., a conventional CPSC system.

This shows that OTFS can be seen as a generalization of both OFDM and CPSC systems.

A. Relation between OTFS and A-OFDM

Now we are ready to reveal the relation between OTFS and A-OFDM systems in static multipath channels.

Specifically, at the transmitter of an A-OFDM system [1, Fig. 1], the input data of length N_c is arranged into an $M \times N$ matrix and a N-point IFFT is then applied to each row. The transmitted outputs after IFFT are read out column-wise and can be written as

$$\check{\mathbf{S}} = \mathbf{X}\mathbf{F}_N^{\mathrm{H}}, \tag{6}$$

which yields the transmitted time domain signal, $\check{\mathbf{s}} = (\mathbf{F}_N^{\mathrm{H}} \otimes \mathbf{I}_M)\mathbf{x}$.

At the receiver of A-OFDM, the N_c received signals are converted to a $M \times N$ matrix and a N-point FFT is applied to each row. Similar to the transmitter, the receiver output of A-OFDM can be written as

$$\begin{aligned} \check{\mathbf{Y}} &= \mathbf{R}\mathbf{F}_N \\ \check{\mathbf{y}} &= (\mathbf{F}_N \otimes \mathbf{I}_M)\check{\mathbf{r}} \end{aligned} \tag{7}$$

Therefore, from (1), (4), (6), and (7), we can conclude that, under static multipath channels, OTFS and A-OFDM systems share the same transmitter and receiver structure (see Fig. 1).

Note that OTFS/A-OFDM uses M copies of an N-point IFFT and FFT at transmitter and receiver, respectively. Comparing to a conventional OFDM with $N_c = MN$ subcarriers, the complexity of OTFS/A-OFDM reduces from $MN\log_2(MN)$ to $MN\log_2 N$ complex multiplications and the maximum PAPR reduces from MN to N.

III. Detection of OTFS/A-OFDM

In this section, we first review traditional ZF and MMSE detections, originally proposed for A-OFDM in [1], [2] and also applicable for OTFS in static multipath channels. Further, we derive a necessary and sufficient condition on the number of subcarriers in OTFS to guarantee that all the transmitted symbols experience uniform channel gains, as in CPSC (a special case of OTFS/A-OFDM). We then apply the low-complexity message passing (MP) detection algorithm for OTFS (see [10]) with improved error performance over ZF and MMSE detections.

A. ZF detection

It was identified in [1, Theorem 1] for A-OFDM that the effective channel matrix has a block diagonal structure, $\mathbf{H}_{\text{eff}} = \text{diag}[\check{\mathbf{H}}_0, \check{\mathbf{H}}_1, \cdots, \check{\mathbf{H}}_{N-1}]$ with $\check{\mathbf{H}}_0, \cdots, \check{\mathbf{H}}_{N-1} \in \mathbb{C}^{M \times M}$. Further, each $\check{\mathbf{H}}_n$, for $n = 0, \cdots, N-1$, can be diagonalized as $\check{\mathbf{H}}_n = \mathbf{F}_M^{\mathrm{H}} \mathbf{D}_n \mathbf{F}_M$. Therefore, from (5), received symbols can be simplified as

$$\mathbf{y}_n = \check{\mathbf{H}}_n \mathbf{x}_n + \widetilde{\mathbf{w}}_n \tag{8}$$

$$\hat{\mathbf{y}}_n = \mathbf{F}_M \mathbf{y}_n = \mathbf{D}_n \mathbf{F}_M \mathbf{x}_n + \mathbf{F}_M \widetilde{\mathbf{w}}_n \tag{9}$$

for $n = 0, 1, \cdots, N-1$. Here, $\mathbf{y}_n$, $\mathbf{x}_n$, and $\widetilde{\mathbf{w}}_n$, are the subvectors formed by taking nM to $(n+1)M-1$ elements from $\mathbf{y}$, $\mathbf{x}$, and $\widetilde{\mathbf{w}}$, respectively. Hence, the estimated symbols after ZF detection can be written as

$$\hat{\mathbf{x}}_n = \mathbf{F}_M^{\mathrm{H}} \mathbf{D}_n^{-1} \hat{\mathbf{y}}_n \tag{10}$$

B. MMSE detection

From (9), the estimated symbols after MMSE detection can be written as

$$\hat{\mathbf{x}}_n = \mathbf{F}_M^{\mathrm{H}} \mathbf{D}_n^{\mathrm{H}} \left(\mathbf{D}_n \mathbf{D}_n^{\mathrm{H}} + \frac{\sigma^2}{E_s} \mathbf{I}_M \right)^{-1} \hat{\mathbf{y}}_n \tag{11}$$

Note that the complexity of ZF and MMSE detectors is of the order of $O(M\log_2 M)$. However, ZF and MMSE linear detectors do not fully exploit the available system diversity. Finally, these detection methods do not take advantage of the sparsity of $\check{\mathbf{H}}_n$.

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C. Message passing detection

For OTFS in static multipath channels, we first establish the relation between $\mathbf{H}_{\rm eff}$ and $\mathbf{H}$ using the following lemma, which is based upon the observation that $\mathbf{H} = {\rm circ}[\mathbf{H}_0, \mathbf{H}_1, \cdots, \mathbf{H}_{N-1}]$, is a block circulant matrix, where $\mathbf{H}_n, 0 \leq n \leq N-1$, are $M \times M$ submatrix.

Lemma 1: $\check{\mathbf{H}}_n(i,j), 0 \leq i,j \leq N-1$, is equal to the n^{th} element in the FFT of $\mathbf{u}_{(i,j)} \triangleq [\mathbf{H}_0(i,j), \cdots, \mathbf{H}_{N-1}(i,j)]$.

Proof: Since $\mathbf{H}$ is a block circulant matrix of N blocks of size $M \times M$, it can be block-diagonalized using $(\mathbf{F}_N \otimes \mathbf{I}_M)$ and $(\mathbf{F}_N^{\rm H} \otimes \mathbf{I}_M)$ [8], [19], and the result follows from (5). ■

Next, using Lemma 1, we prove the following theorem on the minimum value of M in OTFS to guarantee that all the transmitted symbols experience uniform channel gains, as in CPSC.

Theorem 1: The input–output relation in an OTFS system of NM transmitted symbols is equivalent to N parallel CPSCs of length M with the identical time-domain channel, except for an additional phase shift, if and only if $M \geq L$.

Proof:

(If) – For $M \geq L$, the entries of the vectors $\mathbf{u}_{(i,j)}$, $0 \leq i,j \leq M-1$, become

$$\mathbf{u}_{(i,j)} = \begin{cases} [h_{i-j}, 0, \cdots, 0] & \text{if } 0 \leq (i-j) \leq (L-1) \\ [0, h_{L+(i-j)}, \cdots, 0] & \text{if } -(L-1) \leq (i-j) < 0 \\ [0,\ 0,\ \cdots, 0] & \text{otherwise} \end{cases}$$

Taking the FFT's of the above we have

$$\mathbf{v}_{(i,j)} = \begin{cases} [h_{i-j}, h_{i-j}, \cdots, h_{i-j}] & \text{if } 0 \leq (i-j) \leq (L-1) \\ h_{L+(i-j)} \cdot [e^{-j2\pi\frac{0}{N}}, e^{-j2\pi\frac{1}{N}}, \cdots, e^{-j2\pi\frac{N-1}{N}}] & \\ & \text{if } -(L-1) \leq (i-j) < 0 \\ [0, 0, \cdots, 0] & \text{otherwise} \end{cases} \tag{12}$$

According to Lemma 1 and (12), we obtain

$$\check{\mathbf{H}}_n = \begin{bmatrix} h_0 & 0 & \cdots & h_1 e^{-j2\pi\frac{n}{N}} \\ h_1 & h_0 & \cdots & h_2 e^{-j2\pi\frac{n}{N}} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & h_1 & h_0 \end{bmatrix} \tag{13}$$

Hence, from (8) and (13), we can conclude that OTFS input-output relation is equivalent to N parallel CPSCs of length M over identical channels, except for an additional phase shift $e^{-j2\pi\frac{n}{N}}$. Moreover, there are only P non-zero entries in each row and column of $\check{\mathbf{H}}_n$.

(Only if) – For $M < L$, we can easily see that $\mathbf{u}_{(i,j)}$ has at least two non-zero entries for some $0 \leq i,j \leq M-1$. For example, for $M = 2$, the value of $\check{\mathbf{H}}_n$ becomes

$$\check{\mathbf{H}}_n = \begin{bmatrix} \mathbf{v}_{(0,0)}(n) & \mathbf{v}_{(1,0)}(n) e^{-j2\pi\frac{n}{N}} \\ \mathbf{v}_{(1,0)}(n) & \mathbf{v}_{(0,0)}(n) \end{bmatrix}$$

where, $\mathbf{v}_{(0,0)}$ and $\mathbf{v}_{(1,0)}$ are the FFT's of $\mathbf{u}_{(0,0)} = [h_0, h_2, \cdots, 0]$ and $\mathbf{u}_{(1,0)} = [h_1, h_3, \cdots, 0]$, respectively.

Therefore, due to FFT operation, the entries of $\check{\mathbf{H}}_n$ differ in both amplitude and phase for each n, and lower gain channels effect the overall system performance (similar to OFDM). ■

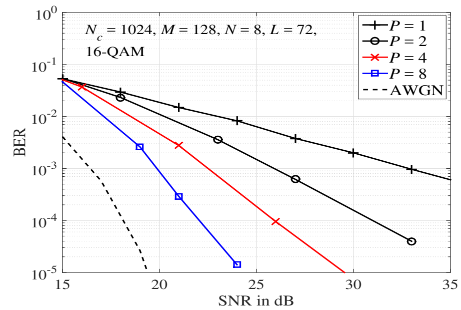

Fig. 2. BER of OTFS for different P with $N_c = 1024$, $M = 128$, $N = 8$, $L = 72$, and 16-QAM

Detector: Since the OTFS input–output relation for static multipath channels is sparse (13), we propose to use the MP algorithm presented in our earlier works [9], [10]. The complexity of MP algorithm for each block is $O(n_{\rm iter} MPQ)$, where $n_{\rm iter}$ is the number of iterations in MP and Q is the modulation alphabet size. In general, even the value of L is large, but the value of P can be small, for example, in LTE vehicular (EVA) channel model, $L = 72$ and $P = 9$.

Remark 1: OTFS has the same performance and detection complexity as N consecutive blocks of CPSCs of length M, but has higher spectral efficiency, since OTFS only requires one CP, whereas CPSC requires N CPs. On the other hand, OTFS has a higher PAPR $= N$ than CPSC (PAPR $= 1$). Therefore, OTFS offers a tradeoff between spectral efficiency and PAPR.

D. Channel estimation

We now propose an embedded pilot channel estimation method to estimate the P non-zero channel coefficients for OTFS with $M \geq L$. In this method, we allocate first M symbols of $\mathbf{x}$ as a header and the remaining $M(N-1)$ symbols for data. In the header, we transmit a known pilot symbol x_p followed by $M-1$ zeros. Therefore, from (8) and (13), $\mathbf{y}_0$ reduces to

$$\mathbf{y}_0(m) = h_m x_p + \widetilde{\mathbf{w}}_0(m) \qquad \text{for } 0 \leq m \leq M-1 \tag{14}$$

and h_m can be estimated using the threshold method proposed in [16], [17]. Note that the pilot power $|x_p|^2$ can be M times higher than the data signal power without increasing the average transmitted power.

Also note that OTFS enables simple correction of any carrier frequency offset (CFO). This is due to the fact that the CFO effect is equivalent to applying a single Doppler shift to all the paths in the OTFS channel [8], [9]. This can be easily detected and corrected in the channel estimation using a pilot signal and thus enables to compensate for much larger CFOs than OFDM.

IV. Numerical results and discussions

In this section, we compare BER of OTFS with OFDM and CPSC for different P and M. In all simulations, we consider $N_c = 1024$ and 16-QAM modulation alphabet ($Q = 16$). In order to obtain BER, we consider 10^5 different channel realizations in Monte-Carlo simulations.

Fig. 2 illustrates the BER performance of OTFS for different $P = 1$, 2, 4, and 8 with $M = 128$, $N = 8$, $L = 72$. Note that we consider $M > L$ in the figure so that all transmitted

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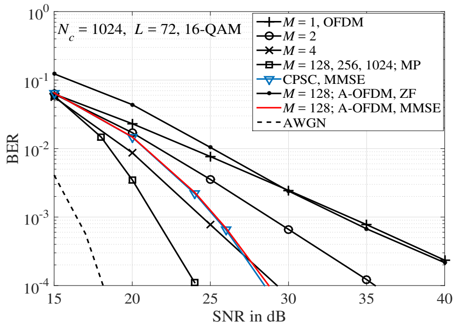

Fig. 3. BER of OTFS for different M with $N_c = 1024$, $L = 72$, and 16-QAM

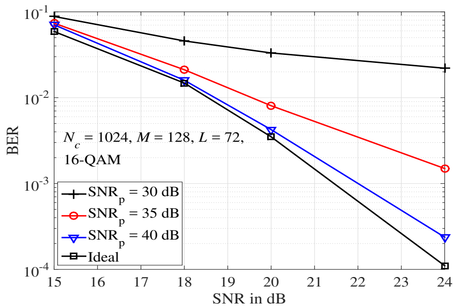

Fig. 4. BER of OTFS for different SNR_p with $N_c = 1024$, $M = 128$, $L = 72$, and 16-QAM

symbols experience equal channel gains. We assume that the P paths are uniformly distributed in L, for example, when $P = 4$, we assume only h_0, h_{23}, h_{46}, and h_{69} have non-zero coefficients. Moreover, if $P = 1$ then it reduces to a flat fading channel. The channel coefficients of the P paths are generated using i.i.d. complex Gaussian distribution, $\mathcal{CN}(0, 1/P)$. Here we adopt MP detection algorithm and assume perfect channel state information (CSI) is available at the receiver. We observe from Fig. 2 that as P increases, the BER slope improves. This diversity advantage is due to the fact that each information symbol experience the channel gains from P paths.

In Fig. 3, we present the performance of OTFS for different M with $L = 72$ and $P = 9$. We consider LTE EVA channel model for generating channel tap coefficients (h_l) and assume perfect CSI is available at the receiver.. We observe that, for $M = 128, 256$, $(M > L)$, the performance of OTFS using MP detection improves with M and achieves the performance similar to CPSC of $M = 1024$, which agrees to Theorem 1. Moreover, OTFS using MP detection outperforms OTFS/A-OFDM using MMSE detection by approximately 5 dB, and OFDM by 15 dB [1]. This is due to the fact that MP detection is approximate to maximum likelihood detection and better exploits the full channel diversity, when compared to MMSE.

Fig. 4 compares the BER of OTFS for different pilot SNRs, $\text{SNR}_\text{p} = |x_p|^2/\sigma^2$, with $M = 128, N = 8$ and $L = 72$. We adopt a threshold of 3σ. We observe that BER performance improves as SNR_p increases and approaches the performance of the perfect CSI (ideal) case for $\text{SNR}_\text{p} = 40$ dB.

V. Conclusion

In this letter, we have studied an $M \times N$ OTFS in static multipath channels and showed that its structure is equivalent to A-OFDM, a scheme proposed in [1]. Further, we have derived a necessary and sufficient condition ($M \geq L$) in OTFS to guarantee that all the transmitted symbols experience uniform channel gains, as in CPSC. We apply a low-complexity MP detection algorithm to OTFS and show that OTFS with MP detection performs similarly to CPSC, but better than OTFS/A-OFDM with ZF and MMSE detections. We also show that the performance of OTFS using channel estimation with embedded pilots approaches the performance with ideal channel state information at the receiver.

[1]We observe very similar behavior of OTFS for other modulation schemes (BPSK, 4-QAM), which are not reported due to lack of space.

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