Rotated multidimensional QAM constellations for the Rayleigh fading channel

Joseph Boutros and Emanuele Viterbo

E.N.S.T, Paris, France
and
Politecnico di Torino, Torino, Italy

Presentation outline

Fading channel

Rayleigh fading channel with perfect phase recovery and coherent demodulation

$$y = \alpha x + n$$

- x: complex transmitted symbol
- n: complex additive white Gaussian noise
- α: real Rayleigh distributed fading amplitude with $E[\alpha^2] = 1$
- y: received complex symbol

With perfect interleaving we can assume independent fading coefficients.

In case of a *deep fade* (small α) the probability of a detection error is large.

Diversity

If we supply the receiver with several replicas of the same information signal over L independently fading channels the probability that all components fade simultaneously is reduced from p to p^L.

- Time diversity
- Frequency diversity
- Space diversity

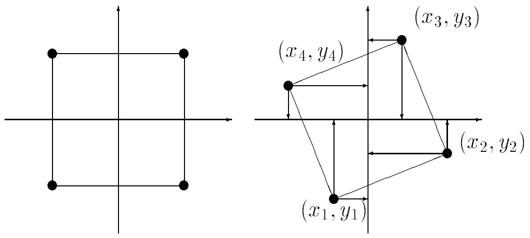
These types of diversity can be viewed as a form of repetition coding. More efficient error control coding techniques can be used to obtain higher diversity orders.

- Code diversity $L = d_{H,min}$

$\oplus$ It can be proved theoretically that for large values of diversity the Rayleigh fading channel acts as an AWGN channel.
$\ominus$ All the above techniques increase diversity at the expense of power, bandwidth or antenna complexity.

How to increase diversity: a simple example

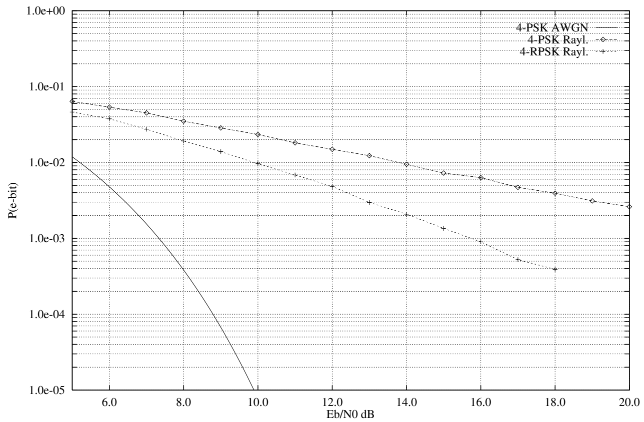
(a) 4-PSK – L=1 (b) 4-RPSK – L=2



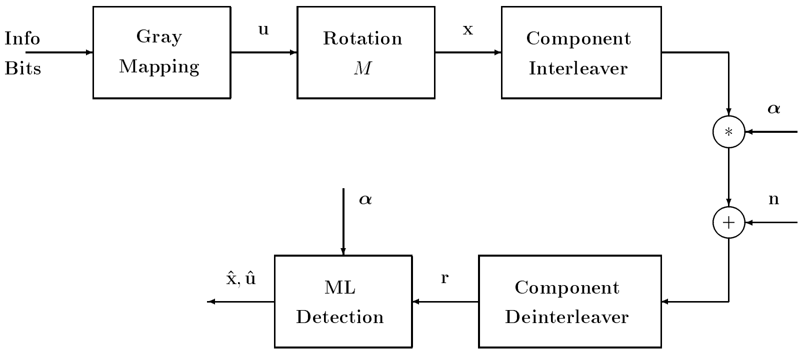
The **modulation diversity** L is equal to the minimum number of distinct components between any two constellation points.

The **product distance** is defined as $d_p(P_i, P_j) = |x_i - x_j||y_i - y_j|$.

Bit error rate 4-PSK vs. 4-RPSK



System model



This system is uncoded.

- $\mathbf{u} = (u_1, \dots u_n)$ $u_i = \pm 1, \pm 3, \dots$ is an n-dimensional QAM signal point and is obtained by grouping $n/2$ QAM complex symbols.
- Transmitted vector $\mathbf{x} = \mathbf{u}M$.
- Received vector $\mathbf{r} = \boldsymbol{\alpha} * \mathbf{x} + \mathbf{n}$.
- $\mathbf{n} = (n_1, n_2, \dots n_n)$ is a noise vector, whose real components n_i are zero mean, N_0 variance Gaussian distributed independent random variables.
- $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots \alpha_n)$ are the random fading coefficients with unit second moment and $*$ represents the component-wise product.
- For ML detection with perfect CSI we minimize the metric

$$m(\mathbf{x}|\mathbf{r}, \boldsymbol{\alpha}) = \sum_{i=1}^{n} |r_i - \alpha_i x_i|^2$$

Problem statement

Chernoff bound on the pairwise error probability

$$P(\mathbf{x} \to \mathbf{y}) \leq \prod_{x_i \neq y_i}^{(L)} \frac{8N_0}{(x_i - y_i)^2}$$

To minimize the union bound to the error probability $\Sigma_{\mathbf{x} \neq \mathbf{y}} P(\mathbf{x} \to \mathbf{y})$ we must:

1. Minimize the average energy per constellation point.
2. Maximize the diversity L.
3. Maximize the minimum L-product distance $d_{p,min}^{(L)} = \Pi_{x_i \neq y_i}^{(L)} |x_i - y_i|$.
4. Minimize the kissing number τ_p for the L-product distance i.e., the total number of points at the minimum L-product distance.

Tackling the problem

- To increase the modulation diversity we must consider high-dimensional rotated constellations.
- Multidimensional QAM constellations can be viewed as a subset carved from a scaled and translated cubic lattice.
- We can use the *Universal Lattice Decoder* for ML detection.
- Algebraic number theory is the mathematical tool which enables to construct rotated lattices with high diversity and good $d_{p,min}$.
- It is possible to design n-dimensional QAM constellations with diversity $L = n/2$ and $L = n$ for some particular values of n.

Algebraic number theory

- An **algebraic number field** $K = \mathbf{Q}(\theta)$ is the set of all possible algebraic combinations $(+, -, *, /)$ of an irrational number θ with the rational numbers of $\mathbf{Q}$.
- From elementary calculus we know that $\mathbf{Q}$ is dense in $\mathbf{R}$, the set of real numbers, then we could state that the set K is "a little bit denser" in $\mathbf{R}$.
- Using the **canonical embedding** it is possible to uniquely represent each element of an algebraic number field with a point in an n-dimensional Euclidean space $\mathbf{R}^n$ just like we represent the elements of $\mathbf{Q}$ in $\mathbf{R}$.

Algebraic number theory

- This set of points is now only dense in $\mathbf{R}^n$ as $\mathbf{Q}$ was in $\mathbf{R}$. In fact we chose n so as to satisfy this condition. n is called the degree of the algebraic number field.
- Within $\mathbf{Q}$ we find the set of relative integers $\mathbf{Z}$ which can be represented as a one dimensional lattice Z in $\mathbf{R}$. In K there exists a subset O_K which corresponds to an n-dimensional lattice $\mathbf{R}^n$.
- The canonical embedding maps O_K into a lattice with a diversity between $n/2$ and n.
- In the case $L = n$ also the minimum product distances are related to the properties of the number field.

The direct approach

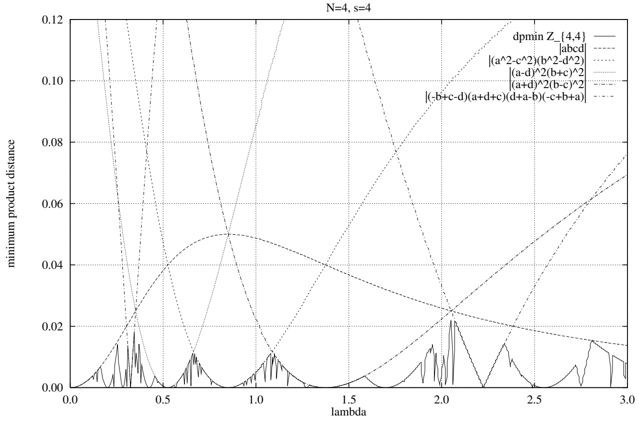
We try to build a rotation matrix M which produces a constellation with full diversity ($L = n$) and maximum $d_{p,min}$. M must satisfy the following constraints

- $\det(M) = 1$
- The row vectors of M are orthonormal.

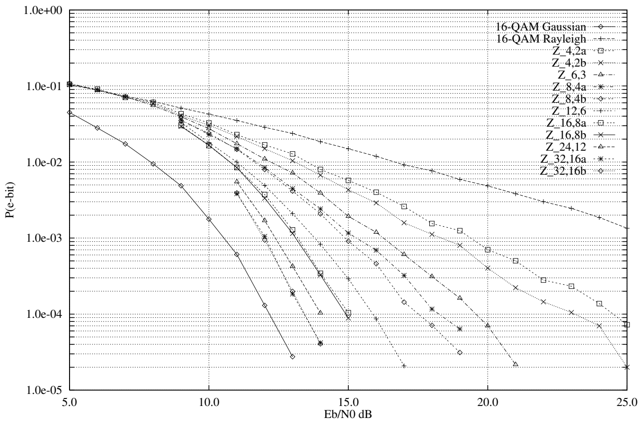
This constraint opimization problem has too many unknowns for a totally arbitrary matrix. We restrict ourselves to some special families of matrices.

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ b & c & a \\ -c & -a & -b \end{pmatrix} \quad \begin{pmatrix} a & b & -c & -d \\ -b & a & d & -c \\ c & d & a & b \\ -d & c & -b & a \end{pmatrix} \quad \begin{pmatrix} a & b & c & -d & -e & -f \\ b & c & a & -e & -f & -d \\ -c & -a & -b & f & d & e \\ d & e & f & a & b & c \\ e & f & d & b & c & a \\ -f & -d & -e & -c & -a & -b \end{pmatrix} \quad \ldots$$

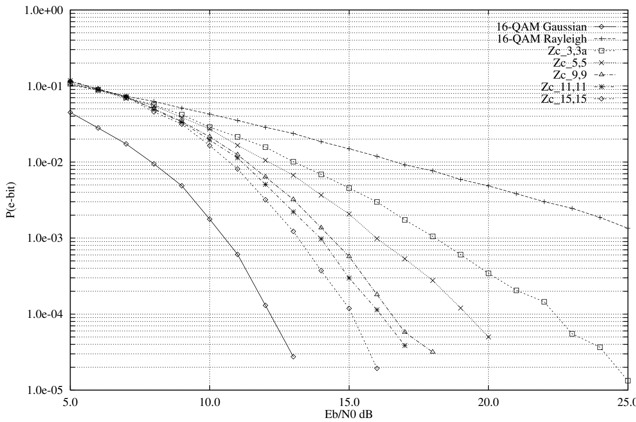
$d_{p,min}$ for a family of rotated Z_4 lattices



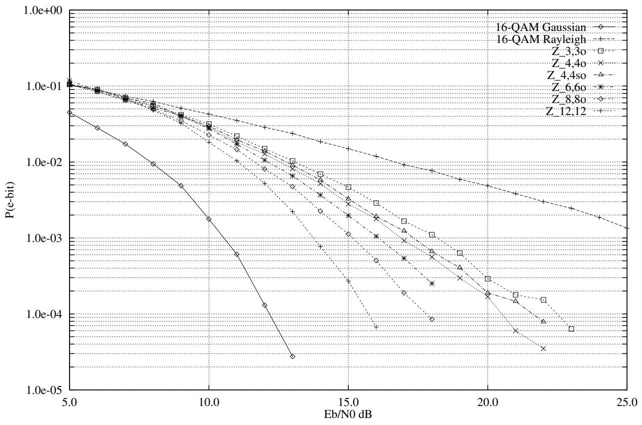
BER for $L = n/2$, $n = 2^{e_1}3^{e_2}$, 4 bits/symbol



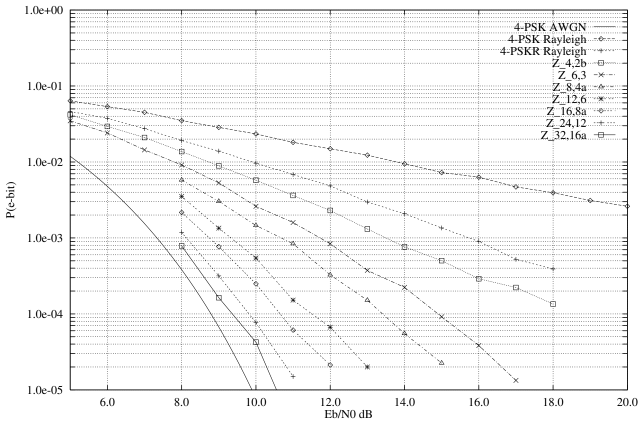
BER for $L = n$, $n = 3, 5, 9, 11, 15$, 4 bits/symbol



BER for $L = n$, $n = 3, 4, 6, 8, 12$, 4 bits/symbol



BER for $L = n/2$, $n = 2^{e_1}3^{e_2}$, 2 bits/symbol



Conclusions

- High diversity implies almost Gaussian performance over the fading channel.
- Diversity is traded for higher demodulator complexity only.
- Maximizing L and $d_{p,min}$ is not enough, τ_p is also very important.
- Using the Universal Lattice Decoder the ML detection complexity is independent of the spectral efficiency. Only increasing the dimension slows down ML detection.
- Future work:
  1. Additional coding schemes over the uncoded system
  2. Effects of imperfect CSI estimation
  3. Decoder implementation problems
  4. Performance analysis with correlated fading channels